

# Approximations to the Lévy LIBOR Model

by

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*“It always seems impossible until it’s done.”*

Nelson Mandela

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# Abstract

In this thesis, we study the LIBOR Market Model and the Lévy-LIBOR. We first look at the construction of LIBOR Market Model (LMM) and address the major problems associated with specifically the drift component of LMM. Due to the complexity of the drift for LMM, the Monte Carlo method seems to be the ideal tool to use. However, the Monte Carlo method is time consuming and therefore an expensive tool to use. To improve on the process we look beyond the dynamics of the lognormal distribution, where Brownian motion (the only Lévy process with continuous paths), is the driving process and apply other Lévy processes with jumps as the driving process in the dynamics of LIBOR. The resulting process is called Lévy LIBOR Model constructed in the framework of [Eberlein and Özkan \(2005\)](#). The Lévy LIBOR model is a very flexible and a general process to use but has a complicated drift part in the terminal measure. The complicated drift term has random terms in the drift part as a result of change of measure. We employ Picard approximation and cumulant expansions in the resulting drift component to make the processes tractable in the framework of [Papapantoleon and Skovmand \(2010\)](#).

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# 1. Introduction

Banks run on the system of time value of money. Mostly if you have an amount of money in your bank account today, that money may not have the same value a year from now. Money can either lose value or gain value depending on the market dynamics. More often the later is true. Banks on the other hand need a smooth flow of currency to meet the demands of their clients all the time. Mostly they do so by taking deposits and also borrowing from the federal bank. They can also borrow from other banks with excess money. When the banks lend money, they charge an interest rate to cater for future fluctuations in the currency. Borrowing from other banks is more expensive than from the central banks, but banks do it anyway because it is more convenient and faster.

Trading between banks grew with an increase in derivative trades such as interest rate swaps, currency options and others that are based on currency. Gradually when more banks joined this faster means of borrowing as well as other currency trade mentioned, some banks charged higher rates whilst others charged low so there was no consistency in the rates. There was the urgency for all the member banks involved in this trade to arrive at a consensus for a bench mark rate that is fair to all. This and many other factors gave birth to the London Interbank Offered Rate (LIBOR) in 1986.

LIBOR is an indicative average interest rate at which a selection of banks (the panel banks)<sup>1</sup> are prepared to lend one another unsecured funds<sup>2</sup> on the London Money Market. Originally LIBOR was calculated for three (3) currencies, US dollar, British Pound sterling and the Japanese Yen. Currently it is being calculated for ten (10) major currencies in the market. The other seven currencies are the Australian dollar, Canadian dollar, Swiss franc, Danish krone, Euro, New Zealand dollar and Swedish krona. LIBOR rates are calculated for 15 borrowing periods and are published daily at 11:30 am (London time) by Thomson Reuters.

Many of financial derivatives and products are tied to LIBOR, hence a very influential rate in the Financial Market. An increase in LIBOR means that banks do not have confidence in each other, so higher rates are applied. Higher rates means higher cost of borrowing and people are discouraged from borrowing. The consequence is the economy does not grow as expected, because spending is minimized. The opposite happens when LIBOR is reduced. A reduced LIBOR is tantamount to more spending and hence growth in the economy.

Some of the major market crashes are the Asian financial crisis in 1997, the bursting of the dot-com bubble in 2000 the subprime mortgage crisis in 2007, the bankruptcy of the Lehman Brothers in 2008 and many others. The occurrences of these events require improving the existing models to capture the reality of the financial Market. To do this, researchers have embarked on making the existing models flexible to reflect the major and minor financial extremes and make the models statistically reliable. The reality is these extremes cannot be eliminated, but they can be managed to some extent (see [Rachev et al. \(2011\)](#)) to make mathematical models more realistic.

Also researchers argue the robust nature of financial models in itself is a major contributor of financial crises. This is as a result of the fact that the financial models used by financial practitioners do not reflect the reality of the financial market. Sometimes the theoretical assumptions are non-realistic, yet they are used to price derivatives and also predict the market. The ideal way to avoid these issues of non-realistic models is to model with tools that try to capture the real market

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<sup>1</sup>The major Banks that give their rates for the calculation of LIBOR

<sup>2</sup>Loans without collateral security

dynamics, since a perfect modelling of the dynamics of the market is also non-realistic.

Mostly asset prices like stocks, bonds and securities fluctuate unexpectedly. The manner in which the prices change in time is not continuous. For example a stock price that costs R50 today might jump to R90 tomorrow due to a lot of market forces. This is what we call a jump in the prices of asset. In modelling when we capture this scenario using Brownian motion, which is a special case of Lévy process with continuous parts, we lose a lot of the reality in the market. The ideal tool to use is discontinuous Lévy processes because they help capture the jump component in the model (see [Tankov \(2009\)](#)).

Models driven by Lévy framework are preferred to the classical diffusion models because they give a better understanding of what goes on in the real market. In terms of pricing of derivatives, they are seen to be the ideal option to resort to. “Models driven by Lévy processes are attractive because of their greater flexibility compared to classical diffusion models ” as in [Eberlein and Özkan \(2005\)](#).

LIBOR as a benchmark interest rate has the characteristic of fluctuating each and every day. These small and large moves fail to follow a continuous path as in the Brownian motion case. An attempt to mimic these fluctuations to reflect the reality makes jump Lévy processes the ideal tool to apply to LIBOR. The introduction of jump processes that reflect those changes makes the models more realistic to the financial market.

“From the risk management point of view, jumps allow to quantify and take into account the risk of strong stock price movements over short intervals, which appears non-existent in the diffusion framework” [Tankov \(2007\)](#).

Over the past years, there have been studies on the use of Lévy models applied to the financial market. In this thesis we use the idea of jump Lévy models that are applied to the LIBOR model. We believe the introduction of jumps will better reflect the real market dynamics.

The flexibility introduced by Lévy LIBOR model is coupled with an apparent pitfall of the corresponding drift component. The terminal drift, grows as a function of the tenor structure and is hard to control, hence makes the process of Lévy LIBOR process intractable. We introduce Picard approximation in the drift component to aid in the tractability of the processes as done in the framework of [Papantoleon and Skovmand \(2010\)](#).

**1.0.1 Organisation of Work.** In what follows, we give a summary of each chapter, the motive behind the tools we adopt and also the hierarchy that this thesis will follow.

In chapter [2](#), we discuss the literature review of Modelling LIBOR and Lévy LIBOR model. Here we mention some of the underlying processes used in the line of research for modelling LIBOR, from Brownian motion, diffusion processes, time homogeneous Lévy processes, time inhomogeneous Lévy processes, general semimartingale processes and affine models.

Chapter [3](#), we introduce basic stochastic calculus that will be used throughout this thesis. We give Basic definitions of probability theory, stochastic processes and results on stochastic calculus that we employ in the main body of the thesis.

In chapter [4](#), the basic theory of Bond market is reviewed. We systematically introduce the building blocks of the bond market, starting with zero coupon bonds, Some basic interest rate related instruments are mentioned, such as forward interest rate and forward rate agreement. LIBOR interest rate is introduced and we also show the relationship between LIBOR and the general Bonds in the market. We also derive the equations for the LIBOR model here.

Having established some basic stochastic calculus, the relations between the basic bonds and know what LIBOR is all about, we introduce the theory of LIBOR and derive the LIBOR Market Model (LMM) in chapter 5. In this framework we use the lognormal model to derive LIBOR, where the underlying random processes used in modelling is the Brownian motion. Here we derive the dynamic equation for LIBOR under the terminal measure. We also address the major pitfalls of using the lognormal approach and associated problems with the drift component. We do some path simulations to show how the LMM model behaves when modelled under different Brownian paths or the terminal Brownian path when using the terminal measure.

In chapter 6, we study some general theory of Lévy processes that will be used in the main work. We mention the two types of Lévy processes as the homogeneous and non-homogeneous types. We emphasize the generalized Lévy processes called Non-homogeneous Lévy processes, when the parameters are time dependent since it comes with more model flexibility compared to constant parameters in the case of the homogeneous type. The main reason for the flexibility of the Non-homogeneous Lévy process is because the stationarity property is broken.

Chapter 7 is the introduction of Lévy LIBOR Market Model. Here we have established the theories of LIBOR and Lévy in the previous two chapters, so we embed the Lévy with LIBOR Market Model to get what we call the Lévy LIBOR Market Model (LLMM) in the framework of Eberlein and Özkan (2005). So instead of considering the lognormal framework, we model LIBOR with Lévy process and this has a great deal of advantages as will be discussed in this chapter. Also the intractability of the terminal drift will be discussed.

Once we have constructed the Lévy LIBOR Model, we study the advantages as being flexible and reflecting the reality of the financial market as compared to the Lognormal framework of LIBOR with Brownian motion as the underlying process. The pitfalls of LLMM are related to the drift and the compensator. The drift term is non-tractable and makes it difficult to track the process. Mostly in the literature the freezing of the drift method is used to truncate the drift. In this method the initial values of LIBOR are used in the drift term to remove the randomness. We will discuss why this process is not the best method to use. The main purpose of chapter 8 is to use an approximation technique to truncate the drift to be tractable, that way we can track the whole processes for Lévy-LIBOR. Here we adopt Picard approximation technique for the approximation of the drift in the framework of Papapantoleon and Skovmand (2010). We also adopt a cumulant expansion to approximate the integral sign in terminal drift as it time consuming when simulating using numerical integration at each step instead of summation.

In Chapter 9, we consider the numerical approximation of Lévy LIBOR model using Picard approximation. Firstly we show the full numerical solution of Lévy LIBOR without using any approximation. We then show the simulation of Lévy LIBOR considering the second order Picard approximation used with numerical integration method. Finally we consider the approximation method when we have considered the second order approximation with both the first order and the second order cumulant expansion, This simulation seems to be the fastest among the three methods considered. Finally we consider the model simulation when we substitute Brownian motion as the jump component of the approximate Lévy LIBOR Model instead of the pure jump NIG process used.

Finally chapter 10 is the conclusion and summary of what has been done. We also give suggestion of what can be considered in further work in the same line of research.

## 2. Review of Relevant Literature

The introduction of LIBOR has led to a smooth trade in the financial market as a result of common benchmark interest rates. So what are some key questions financial mathematicians will ask regarding LIBOR? The questions that arise are: how representative is LIBOR in terms of the effect of interest rates, does it predict the reality in the market, are the panel banks giving the accurate values for LIBOR? In terms of the theory of LIBOR, are the rates tractable? These questions and many others have led to the introduction of models that seek to model the general term structure of interest rates to fit the real market. This led to the birth of the LIBOR market model (LMM), also known as the BGM Model (Brace Gatarek Musiela Model).

The LIBOR Market Model (LMM) is the first model of interest rates dynamics that is consistent with the market practice of pricing interest rate derivatives. LMM was first introduced by [Brace et al. \(1997\)](#), [Sondermann et al. \(1995\)](#), [Miltersen et al. \(1997\)](#), where Brownian motion is the driving process.

Brownian motion as the driving process for LMM is found to be inconsistent with the Market dynamics, which has led to the introduction of Lévy process which originated in the first half of the last century and these were used in finance only in the last decade of the century. This class of models is flexible and depicts the reality of the Financial Market better than Brownian motion (the only Lévy process with continuous paths).

Models with Lévy processes as the driving process were introduced by [Eberlein and Raible \(1999\)](#). Later the notion of time-inhomogeneous Lévy processes was introduced by [Ernst et al. \(2005\)](#) where the parameters of the Lévy processes are time dependent and is the general case of the Lévy processes with deterministic parameters (time homogeneous). By this generalisation, “the model allows one to accurately capture the empirical dynamics of interest rates, whilst it is still analytically tractable, so that closed form valuation formulas for liquid derivatives can be derived” as shown in [Antonis \(2007\)](#).

Extension of LIBOR Market model using semimartingales is also given in [Jamshidian \(1999\)](#), [Antonis \(2007\)](#), [Papantoleon et al. \(2012\)](#), and [Glasserman and Kou \(2003\)](#). Also in [Özkan \(2002\)](#), we see the Lévy term structure framework embedded in the general semimartingale approach by [Jamshidian \(1999\)](#). LIBOR Market models with affine processes are also developed by [Keller-Ressel et al. \(2009\)](#).

In the following papers stochastic volatility is used as the driving process for LIBOR, [Andersen et al. \(2005\)](#), [Lixin and Fan \(2006\)](#) and [Belomestny et al. \(2009\)](#).

Analytical tractability of the LIBOR Market Models has been discussed in the papers [Joshi and Stacey \(2008b\)](#) and [Gatarek et al. \(2006\)](#). Here the freezing the drift is used as an approximation of the drift component. Another line of research opposes the use of freezing the drift approximation as “crude and does not yield acceptable results” as [Papantoleon and Skovmand \(2011\)](#) put it.

Pricing of Caps and Swaptions in jump diffusion models is tedious. Resorting to approximation schemes, [Glasserman and Merener \(2003a\)](#) and [Glasserman and Merener \(2003b\)](#) have constructed some approximation tools for the pricing of caps and Swaptions in jump diffusion models. In [Eberlein and Özkan \(2005\)](#) and [Klunge \(2005\)](#) valuation methods for caps and floors are also discussed whilst Calibration is dealt with extensively in [Eberlein and Klunge \(2007\)](#).

In another line of research LIBOR rates are calculated by backwards induction with driving process



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as time inhomogenous Lévy processes [Eberlein and Özkan \(2005\)](#). However, the LIBOR rates have different Brownian motions and different Compensators using their respective forward measures. We resort to the terminal measures, where the process is expressed in one Brownian motion (terminal Brownian motion) and one compensator (terminal compensator). In the dynamics of the terminal measures, using the respective terminal Brownian motion and the terminal compensators, the only apparent problem is the complicated terminal drift term. The specification of the drift that makes the process a martingale is complicated as a result of random terms in the drift. The drift term grows as a function of the tenor structure, hence the larger the tenor, the more untraceable the drift component. As a result, we resort to approximation methods to truncate the drift component when the terminal measures are used. The approximation method used here is called Picard approximation. The use of this approximation is central to tracking the drift component and also with an advantage of simulating the LIBOR rates in parallel because. This parallelism is achieved because the respective LIBOR rates depend on a time inhomogeneous Lévy process achieved using the second order Picard iteration and not dependent on the LIBOR rates. The following papers consider approximation of the drift component using Picard approximation [Papantoleon et al. \(2012\)](#), [Papantoleon and Skovmand \(2010\)](#).

### 3. Stochastic Calculus

In this chapter, we discuss and give some important definitions, theorems, lemmas, propositions and proofs that will be used later in this thesis on stochastic calculus. The following books are used for this section and readers can refer to for more details: Lin (2004), Miron and Swannel (1991), Bhattacharya and Waymire (1990), Baxter and Rennie (1996), Etheridge (2002), Korn et al. (2010a), Klebaner (2005), Dieter (2006). Also, we adopt the following symbols in our discussions:

Symbol	Meaning
$\mu$	Drift
$\sigma$	Volatility
$r$	Risk free interest rate
$W$	Brownian motion
$P \ll Q$	P is absolutely continuous with respect to Q
$P \sim Q$	P is equivalent to Q

#### 3.1 Probability Spaces and Filtrations

We start by summarising basic probability theory and information structure as in Lin (2004) and Cont and Tankov (2004): We note that the notations used is the framework of Protter (2004). We consider  $(\Omega, \mathcal{F}, P)$  as the probability space where  $\Omega$  is the sample space,  $\mathcal{F}$  the information structure on  $\Omega$  and  $P$  the probability measure on  $\mathcal{F}$ .  $\mathbb{F}$  is the filtration with increasing family of  $\sigma$ -algebras for  $\mathcal{F}_{0 \leq t \leq \infty}$ , where  $\mathcal{F}_s \subset \mathcal{F}_t$ , for  $s < t$  (Protter (2004)).

**3.1.1 Definition.** A filtered complete probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  is said to satisfy the usual hypotheses if

1.  $\mathcal{F}_0$  contains all the  $P$ -null sets of  $\mathcal{F}$ ,
2.  $\mathcal{F}_{\square} = \bigcap_{u>t} \mathcal{F}_u$ , all  $t, 0 \leq t < \infty$ ; that is, the filtration  $\mathbb{F}$  is right continuous.

Generally with an increase in the information structure we expect the probabilities of the occurrence of random variables to change. *Conditional probabilities* on the dynamics of the filtration are denoted by  $(P | \mathcal{F}_t)$ .

**3.1.2 Definition.** [Adapted Karatzas and Shreve (1998)] A stochastic process  $X$  is adapted to the filtration  $\mathbb{F}$  if for each  $t \geq 0$ ,  $X_t$  is an  $\mathbb{F}_t$  measurable random variable.

#### 3.2 Stochastic Processes

Here we discuss the two major continuous process used as basics in Stochastic processes. The Brownian motion (Wiener process) and Geometric Brownian motion.

**3.2.1 Definition.** [Wiener Process (Brownian Motion) Björk (2004)] A stochastic process  $W$  is called a Wiener process if the following are satisfied:

- (1.)  $W(0)=0$ .
- (2.)  $W$  have stationary increments: when  $s < r$ ,  $W_s - W_r$  has the same distribution as  $W_{r-s}$ .
- (3.) The process  $W$  has independent increments, that is  $r < s \leq t < u$  then  $W(u) - W(t)$  and  $W(s) - W(r)$  are independent stochastic variables.
- (4.) For  $s < t$  the stochastic variable  $W(t) - W(s)$  has the Gaussian distribution  $N[0, t - s]$ .
- (5.)  $W$  has continuous trajectories.

Brownian motion is the first model used for the dynamics of stock price. It was used in 1900 by Louis Bachelier. The major problem associated with the model is the unrealistic negative prices produced as shown in figure 3.1a. Geometric Brownian motion was later introduced in 1961 by Case Sprenkle to address the problem of negative prices as can be seen in the figure 3.2a Bellalah (2008).

**3.2.2 Definition. [Geometric Brownian Motion [Cont and Tankov (2004)]]** A geometric Brownian motion (GBM) (also known as exponential Brownian motion) is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift<sup>1</sup>. The stochastic differential equation for GBM is given as

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where  $\mu$  is the drift and  $\sigma$  is the volatility. The SDE has the following analytic solution

$$S_t = S_0 \exp \left( \left( \mu - \frac{\sigma^2}{2} \right) t + \sigma W_t \right).$$

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<sup>1</sup>Drift is the change in the average value of the stochastic process

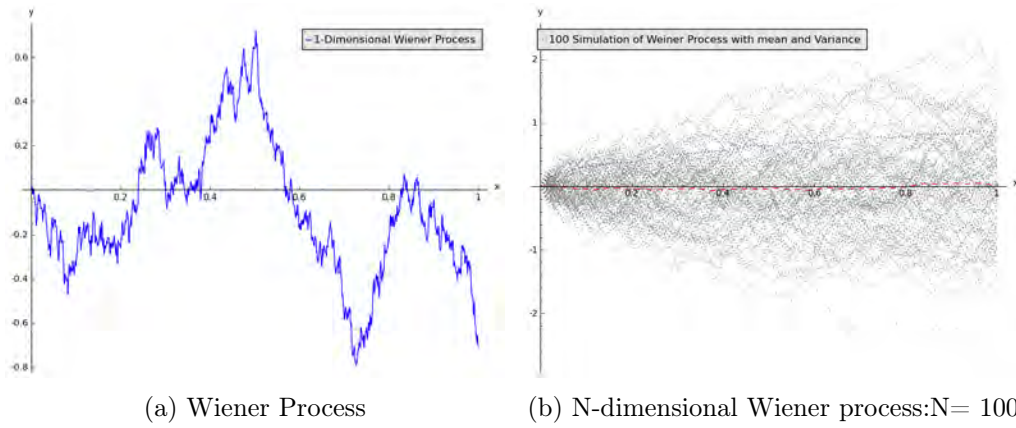


Figure 3.1: Wiener Process

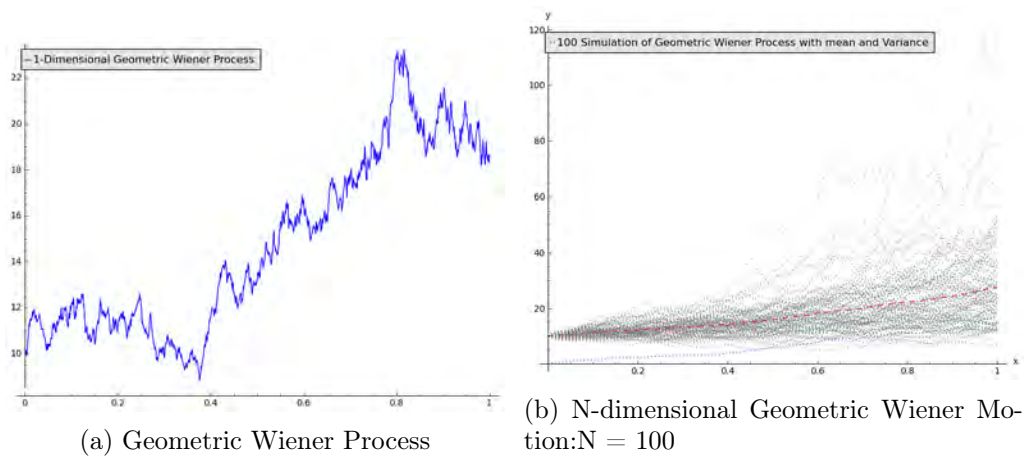
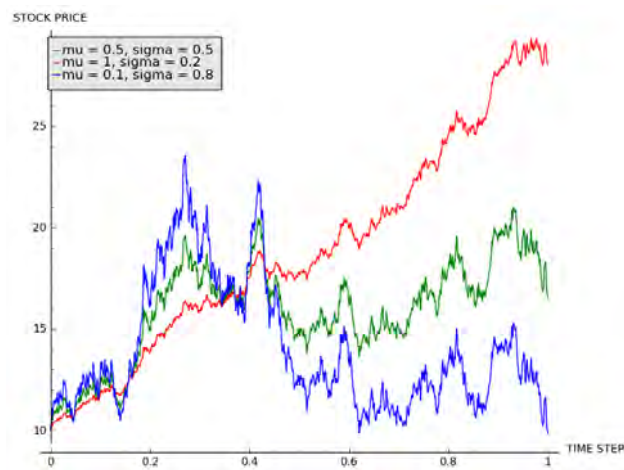


Figure 3.2: Geometric Wiener Process

Figure 3.3: Geometric Process with Different Values of  $\mu$  and  $\sigma$ 

In figure 3.3, we show the dynamics of geometric Brownian motion considering different drifts and volatilities. The blue line shows the dynamics when we have a small drift and a bigger volatility, as

compared to when the drift and variance are the same for the green line. Also the red line shows the dynamics when we have a bigger drift and small volatility. Here we give the formal definition of Stochastic differential equations as follows:

**3.2.3 Definition. [Stochastic Differential Equation (SDE) Haugh (2010)]** An  $n$ -dimensional stochastic differential equation (SDE), has the form

$$X_t = x + \int_0^t \mu(X(s), s)dt + \int_0^t \sigma(X(s), t)dW(s),$$

where the short hand form is

$$dX(t) = \mu(X(t), t)dt + \sigma(X(t), t)dW(t); \quad X_0 = x$$

$W$  is an  $m$ -dimensional standard Brownian motion, and  $\mu$  and  $\sigma$  are  $n$ -dimensional and  $n \times m$ -dimensional  $\mathcal{F}_t$  adapted processes, respectively.

**3.2.4 Theorem. [Itô's Formula Björk (2004)]** Assume the process  $X$  has a stochastic differential given by

$$dX(t) = \mu dt + \sigma dW(t) \tag{3.2.1}$$

where  $\mu$  and  $\sigma$  are adapted processes, and  $f(t, X(t))$  be a  $\mathcal{C}^{1,2}$ -function. Then the stochastic differential of  $f(t, X(t))$  is given by

$$df(t, X(t)) = \left( \mu \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} + \frac{1}{2} \sigma^2 \frac{\partial^2 f}{\partial x^2} \right) dt + \sigma \frac{\partial f}{\partial x} dW(t) \tag{3.2.2}$$

### 3.3 Girsanov's Theorem

This theorem describes the relationship between the stochastic process and the change of measure. In finance it describes how to shift from one probability space to another using the Radon-Nikodym derivative. With this change of measure, the rate of growth in the drift variable changes, but the volatilities remain the same (Hull (2006)). We give the theorem of Radon-Nikodym derivative as shown in Cont and Tankov (2004).

**3.3.1 Theorem. [(Radon-Nikodym theorem) Cont and Tankov (2004)]:** If  $\mu_2 \ll \mu_1$ , then there exists a measurable function  $Z(T) : E \rightarrow [0, \infty]$ , such that for any measurable set  $A$

$$\mu_2(A) = \int_A Z d\mu_1 = \mu_1(Z1_A).$$

The function  $Z(T)$  is called the density or the Radon-Nikodym derivative of  $\mu_2$  with respect to  $\mu_1$  and is denoted by  $\frac{d\mu_2}{d\mu_1}$ . For any  $\mu_2$ -integrable function  $f$

$$\int_E f d\mu_2 = \int_E Z f d\mu_1.$$

Therefore if  $\mu_2$  is absolutely continuous with respect to  $\mu_1$ , an integral with respect to  $\mu_2$  is a weighted integral with respect to  $\mu_1$ , the weight given by  $Z$ .

If  $\mu_2$  is absolutely continuous to  $\mu_1$  and  $\mu_1$  is absolutely continuous with respect to  $\mu_2$ , then  $\mu_1$  and  $\mu_2$  are said to be equivalent measures.

**3.3.2 Definition. [Martingale (Musiel and Rutkowski (2004))]** A real-valued,  $\mathcal{F}$ -adapted process  $M = (M_t)_{t \in [0, T]}$ , defined on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F})_{t \in [0, t]}, P)$ , is called an  $\mathcal{F}$ -martingale with respect to the filtration  $\mathcal{F}$  if the following conditions are satisfied:

- (i.)  $M$  is integrable, that is,  $E_P |M_t| < \infty$  for  $t \in [0, T]$ ,
- (ii.) the following martingale equality holds, for any  $0 \leq t \leq u \leq T$ ,

$$E_P[M_u | \mathcal{F}_t] = M_t \quad (3.3.1)$$

If the equality in equation (3.3.1) is replaced by  $\leq$ , then we say that  $M$  is a supermartingale, and it is called a submartingale if the equality is replaced by  $\geq$ .

**3.3.3 Theorem. [(Girsanov theorem I) Øksendal (2007)]** Let  $X(t)$  be an  $n$ -dimensional Itô process of the form

$$dX(t) = \alpha(t, \omega)dt + \sigma(t, \omega)dW(t); \quad 0 \leq t \leq T,$$

where  $\alpha(t) = \alpha(t, \omega) \in \mathbb{R}^n$ ,  $\sigma(t) = \sigma(t, \omega) \in \mathbb{R}^{n \times m}$  and  $W(t) \in \mathbb{R}^m$ . Assume that there exists a process  $\theta(t) \in \mathbb{R}^m$  such that

$$\sigma(t)\theta(t) = \alpha(t) \quad \text{for almost all } (t, \omega) \in [0, T] \times \Omega$$

and such that the process  $Z(t)$  defined for  $0 \leq t \leq T$  by

$$Z(t) = \exp \left( - \int_0^t \theta(s, \omega) dW_s - \frac{1}{2} \int_0^t \theta^2(s, \omega) ds \right)$$

exists. Let  $Q$  be a measure on  $\mathcal{F}_T$  given by

$$dQ(\omega) = Z(T)dP(\omega)$$

Assume that

$$E_P[Z(T)] = 1.$$

Then  $Q$  is the probability measure on  $\mathcal{F}_T$ ,  $Q$  is equivalent to  $P$  and  $X(t)$  is a local martingale with respect to  $Q$ .

**3.3.4 Definition. [Local martingale Musiel and Rutkowski (2004)]** A process  $M$  is called a local martingale with respect to  $\mathcal{F}$  if there exists an increasing sequence  $(\tau_n)_{n \in \mathbb{N}}$  of stopping times such that  $\tau_n$  tends to  $T$  almost surely, and for every  $n$  the process  $M^n$ , given by the formula

$$u(x) = \begin{cases} M_{t \wedge \tau_n(\omega)}(\omega), & \text{if } \tau_n(\omega) > 0 \\ 0, & \text{if } \tau_n(\omega) = 0 \end{cases}$$

is a uniformly integrable martingale. Any sequence  $(\tau_n)_{n \in \mathbb{N}}$  with these properties is called the *reducing sequence* for a local martingale  $M$ .

**3.3.5 Lemma. [Øksendal and Sulem (2007)]** Suppose that  $Q \ll P$  with  $\frac{dQ}{dP} = Z(t) > 0$  on  $\mathcal{F}_T$ . Let  $X(t)$  be an adapted process with respect to  $(\Omega, \mathcal{F}, P)$  and let  $Z(t)$  be a martingale with respect to  $(\Omega, \mathcal{F}, P)$  such that  $Z(t)X(t)$  is a martingale with respect to  $(\Omega, \mathcal{F}, P)$ . Then  $X(t)$  is a martingale with respect to  $Q$ . Similarly, if  $Z(t)X(t)$  is a local martingale with respect to  $P$ , then  $X(t)$  is a local martingale with respect to  $Q$ .

*Proof.* Here we prove the later statement in the theorem 3.3.5 using the Baye's theorem as in definition ??: Let  $\tau \geq t$  be the time,  $\tau \leq T$ . Then

$$E_Q[X(\tau) | \mathcal{F}_t] = \frac{E_P[Z(T)X(\tau) | \mathcal{F}_t]}{E_P[Z(T) | \mathcal{F}_t]} \quad (3.3.2)$$

$$= \frac{E_P[Z(\tau)X(\tau) | \mathcal{F}_t]}{Z(t)} = \frac{Z(t)X(t)}{Z(t)} \quad (3.3.3)$$

$$= X(t) \quad \text{for all } t \in [0, T] \quad (3.3.4)$$

□

**3.3.6 Proposition.** [*Musiela and Rutkowski (2004)*(page 608)] Let  $W$  be a one-dimensional standard Brownian motion on a probability space  $(\Omega, \mathcal{F}, P)$ . For real number  $\gamma \in \mathbb{R}$ , we define the process  $X$  by setting  $\tilde{W}_t = W_t - \gamma t$  for  $t \in [0, T]$ . Let the probability measure  $Q$ , equivalent to  $P$  on  $(\Omega, \mathcal{F}_T)$ , be defined through the formula

$$\frac{dQ}{dP} = \exp(\gamma W_T - \frac{1}{2}\gamma^2 T)$$

Then  $X$  is a standard Brownian motion on the probability space  $(\Omega, \mathcal{F}, P)$ .

Proposition 3.3.6 can be generalised in the case of stochastic drift term as follows:

**3.3.7 Theorem.** [*Musiela and Rutkowski (2004)*(Page 608)] Let  $W$  be a standard  $d$ -dimensional Brownian motion on a filtered probability space  $(\Omega, \mathcal{F}, P)$ . Suppose that  $\gamma$  is an  $\mathbb{R}^d$ -valued  $\mathbb{F}$ -progressively measurable process such that

$$\mathbb{E}_P \left\{ \mathcal{E}_T \left( \int_0^\cdot \gamma_u \cdot dW_u \right) \right\} = 1. \text{ (where } \mathcal{E} \text{ is defined as stochastic exponential in Proposition 6.2.4)} \quad (3.3.5)$$

Define a probability measure  $Q$  on  $(\Omega, \mathcal{F}_T)$  equivalently to  $P$  by means of the Randon-Nikodym derivative as in theorem 3.3.1

$$\frac{dQ}{dP} = \mathcal{E}_T \left( \int_0^\cdot \gamma_u \cdot dW_u \right) \quad (3.3.6)$$

Then the process  $\tilde{W}$  (Brownian motion for the new measure  $Q$ ) given by the formula

$$\tilde{W}_t = W_t - \int_0^t \gamma_u du, \forall t \in [0, T] \quad (3.3.7)$$

follows a standard 1-dimensional Brownian motion on the space  $(\Omega, \mathcal{F}, Q)$ .

## 3.4 Numeraire

We use the notion of numeraire if we need a standard of measurement. For example the standard of measurement for South African currency is Rand (R). When we measure the standard of goods and services in one specific unit we call it numeraire. Formally it is defined as follows:

**3.4.1 Definition.** [Numeraire *Brigo and Mercurio (2006)*] A numeraire is any positive non-dividend-paying asset.

We consider the savings account  $S(t)$  as our numeraire in measuring the bonds. Given the probability space  $(\Omega, \mathcal{F}, P)$ , a measure  $Q$  on the probability space is an equivalent martingale measure (EMM), for  $S(t)$  if and only if

- $Q \sim P$
- $\tilde{B}(t, T) = \frac{B(t, T)}{S(t)}$  is a  $Q$ -martingale

The measure  $Q$  is also called the risk neutral measure.



## 4. The Bond Market

In this chapter, we introduce the basic definitions that are used in the thesis regarding bonds. The definitions and discussions about bonds used in this chapter can be found in books by Brigo and Mercurio (2006), Musiela and Rutkowski (2004), (Pelsser (2000)), Miron and Swannell (1991), (Vecer (2011)). Here we adopt the following symbols:

Symbol	Meaning
$t$	General time
$\tau$	Present time
$S$	Maturity time
$T$	Expiry time
$r$	Risk-free interest rate
$S(t)$	Savings account
$B(t, T)$	Zero coupon bond
$D(t, T)$	Stochastic discount factor
$f(t, T)$	The instantaneous forward rate
$f_B(\tau, t, T)$	The forward bond price

### 4.1 Basic Bonds

Generally when one borrows  $R100$  from a Bank, one is expected to pay back the same  $R100$  to the Bank plus an interest<sup>1</sup> component  $r$ , which is mostly a percentage of the amount borrowed. The two major types of interest rate are simple (discrete) or continuously compounded rate. In a real market interest accrues discretely according to a given time interval whereas continuously compounding rates are mathematical concepts that are used for modelling purposes. For models to be more realistic, discrete compounding is sometimes used in modelling.

Interest varies from lender to lender. The motive is money loses value (if not invested) with time - the value of the money borrowed today will not be the same as the value in a year from now. One major cause of currency losing value is inflation. In addition, the lender should get some benefit for the risk of undertaking such a transaction.

Borrowing can be individual borrowing from a bank in the form of a loan, or a Bank and a company borrowing from an individual in the form of a bond. A bond is a financial security promising the holder a guaranteed interest at the time of contract expiry and possibly at other times.

The three major dates we will use here are the trade date, effective date and the maturity date. When the trader enters into an agreement and signs a contract to trade for a specific period stipulated in the contract, then this time is called the trade date. Mostly the trade date is not the same as the time when the contract really starts to accrue, called the effective date. Maturity or expiry date is when the contract ceases to exist.

In the terminology of the Bond Market, the writer (seller) is the borrower, whilst the lender (buyer) is the lender. Principal is the initial amount of an investment. Par value or face value is the amount

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<sup>1</sup>Interest is an amount that lenders charge borrowers when they lend

paid to bond holders at maturity. In terms of a coupon bearing bond, face value is the principal plus the coupon payment at the bond maturity date Capiński and Zastawniak (2003).

Mathematically, we denote  $B(t, T)$  as the bond with trade date at time  $t$  which is formally called the present value. At the expiry time  $T$ , we receive  $B(T, T)$ , which is called the face value. We call the type of Bond a zero coupon Bond, if we lend an amount  $B(t, T)$  of a currency at time  $t$ , and receive one unit of the currency at maturity  $T > t$ .

**4.1.1 Definition. [Forward Interest Rate Agreement (FRA)]** FRA starting at time  $t$  and expiring at time  $T$  is a contract, worth nothing at the present time  $\tau$  where  $\tau < t < T$ , but one party agrees to invest principal  $A$  at time  $t$ , for a period  $(T - t)$  and the other party receiving the investment agrees to provide interest at the simple rate  $f(\tau, t, T)$  for the period and then returns principal plus simple interest at time  $T$ . Then

$$\text{Amount Paid} = A(1 + (T - t)f(\tau, t, T)) \quad (4.1.1)$$

In terms of the continuous compounding case we have the following:

$$\text{Amount Paid} = Ae^{r(T-t)} \quad (4.1.2)$$

where  $r$  is the continuously compounded interest rate.

**4.1.2 Definition. [The forward bond price]** The forward price of a bond  $f_B(\tau, t, T)$  worth 0 at present time  $\tau$  is an agreement by one party to sell a bond at time  $t \geq \tau$  for price  $f_B(\tau, t, T)$  which matures at time  $T$  with value 1. The no-arbitrage forward price can be shown to be

$$f_B(\tau, t, T) = \frac{B(\tau, T)}{B(\tau, t)} \quad (4.1.3)$$

By equation (4.1.1), if  $A = 1$  then  $(1 + \delta f(\tau, t, T))$  is the return at time  $T$ , of investing 1 unit at time  $t$ , given a forward rate  $f(\tau, t, T)$ . Investing  $\frac{1}{1 + \delta f(\tau, t, T)}$  at time  $t$ , with this forward rate gives a return of 1 at time  $T$ . Hence we have the following equality from equation (4.1.3)

$$\frac{1}{1 + \delta f(\tau, t, T)} = f_B(\tau, t, T) = \frac{B(\tau, T)}{B(\tau, t)}$$

hence

$$f(\tau, t, T) = \frac{1}{\delta} \left( \frac{1}{f_B(\tau, t, T)} - 1 \right) \quad (4.1.4)$$

**4.1.3 Price of a Zero Coupon Bond. For the simple rate case**

$$1 = (1 + (T - t)r)B(t, T) \quad (4.1.5)$$

$$B(t, T) = \frac{1}{(1 + (T - t)r)} \quad (4.1.6)$$

**For the continuous compounding rate case**

$$1 = e^{(T-t)r} B(t, T) \quad (4.1.7)$$

$$B(t, T) = e^{-(T-t)r} \quad (4.1.8)$$

Hence from equation (4.1.8), we can see the relationship between the bond and the interest rate  $r$ . When the interest rate for a contract is fixed (set at a constant rate for the interval) for a specified interval it is called simple rate. On the other hand when the interest rate is changing depending on the dynamics of the market, it is called a *floating rate* or *stochastic rate*. In practice, floating rate is mostly used because bonds and other derivatives depend on the dynamics of the economy. From equation (4.1.8), floating rate is denoted by

$$r(t, T_i) = \frac{-\log B(t, T_i)}{\delta}, \quad \delta = T_{i+1} - T_i$$

**4.1.4 Forward Interest Rates.** Floating rates fluctuate depending on the state of the market. Parties can agree on a rate before hand that will be applied for the investment within the time interval  $[t, T]$  at time  $\tau$ . This rate is called forward rate  $f(\tau, t, T)$ , where  $\tau$  is the agreement time,  $t$  is the effective time and  $T$  is the maturity of the contract.

$$e^{(T-t)f(\tau, t, T)} = \frac{B(\tau, T)}{B(\tau, t)}$$

Taking the log of both sides of equation (4.1.4) to get

$$f(\tau, t, T)(t - T) = \log B(\tau, T) - \log B(\tau, t) \quad (4.1.9)$$

$$f(\tau, t, T) = \frac{\log B(\tau, T) - \log B(\tau, t)}{(t - T)} \quad (4.1.10)$$

$$= -\frac{\log B(\tau, T) - \log B(\tau, t)}{(T - t)} \quad (4.1.11)$$

$f(\tau, t, T)$  is the *forward rate* for this contract. We find the limit of both sides of equation (4.1.11) as  $T$  approaches  $t$  as

$$\lim_{T \rightarrow t^+} f(\tau; t, T) = -\lim_{T \rightarrow t^+} \frac{1}{B(\tau, T)} \left( \frac{\log B(\tau, t) - \log B(\tau, T)}{T - t} \right) \quad (4.1.12)$$

$$f(\tau, t) = -\frac{\partial \ln B(\tau, t)}{\partial t} \quad (4.1.13)$$

here  $f(\tau, t)$  is called the *instantaneous forward rate*.

Given the instantaneous forward rate  $f(\tau, t)$  we can find the initial price of the Bond at time  $t$ .

$$B(\tau, t) = \exp \left( -\int_{\tau}^t f(\tau, u) du \right) \quad (4.1.14)$$

**4.1.5 Definition. [Bank account (Money-Market account) Brigo and Mercurio (2006)]**

We define  $S(t)$  to be the value of a bank account at time  $t \geq 0$ . We assume  $S(0) = 1$  and that the bank account evolves according to the following differential equation:

$$dS(t) = r(t)S(t)dt, \quad S(0) = 1 \quad (4.1.15)$$

where  $r(t)$  is a positive (possibly stochastic) function of time. As a consequence,

$$S(t) = \exp \left( \int_0^t r(u) du \right) \quad (4.1.16)$$

$r(t)$  is known as instantaneous spot rate or spot rate.

It can be shown that

$$r(t) = f(t, t) = \lim_{T \rightarrow t^+} f(t, T). \quad (4.1.17)$$

To know how the bank account accrues for a given time interval  $\delta$ , we find the first order expansion of equation (4.1.15) to get the following:

$$\begin{aligned} S(t + \delta) &\approx S(t)(1 + \delta r(t)) \\ S(t + \delta) &\approx S(t) + S(t)\delta r(t) \\ \frac{S(t + \delta) - S(t)}{S(t)} &\approx r(t)\delta \end{aligned} \quad (4.1.18)$$

Equation (4.1.18) shows approximately how the bank account accrues at every time step  $\delta$  for a rate  $r(t)$ .

**4.1.6 Definition.** [The stochastic discount factor Musiela and Rutkowski (2004)] The stochastic discount factor  $D(t, T)$  between two time instants  $t$  and  $T$ , is the amount at time  $t$  that is “equivalent” to one unit of currency payable at time  $T$ , and is given by

$$D(t, T) = \frac{S(t)}{S(T)} = \exp \left( - \int_t^T r(u) du \right).$$

**4.1.7 Tenor Structure.** The tenor is the “prespecified collection of settlement dates” in the future specified by

$$0 < T_1 < T_2 < T_3 < \cdots < T_n < T_{n+1}$$

where the  $i$ th tenor length is  $\delta_i = T_{i+1} - T_i$ . We also denote the last time in the tenor structure  $T^* = T_{n+1}$  and  $T_i^* = T^* - i\delta$ .

Here the tenor length is considered as a constant for convenience,  $\delta_i = \delta$  for all  $i$ . In the real Market  $\delta$  is not constant. If  $\delta$  is a month, then it can take 31, 30 days or 28 for February on a normal year and 29 a leap year.

From equation (4.1.4), when we consider two time intervals  $T_1$  and  $T_2$ , then the forward rate is given by

$$f(t, T_1, T_2) = \frac{1}{\delta} \left( \frac{1}{f_B(t, T_1, T_2) - 1} \right) \quad (4.1.19)$$

**4.1.8 LIBOR Interest Rate.** Here we define  $L(T_i)$  as the *LIBOR interest rate* agreed upon at time  $T_i$  on the price of the bond  $B(T_i, T_{i+1})$ , which lasts for the time interval  $\delta = T_{i+1} - T_i$ , such that the the price of the bond is rebalance at any time  $T_i$  to get a yield 1 at the maturity time  $T_{i+1}$ . The mathematical relation for such an agreement is given by

$$1 = (1 + \delta L(T_i))B(T_i, T_{i+1}). \quad (4.1.20)$$

From equation (4.1.20), we solve for  $L(T_i)$  as the LIBOR Interest rate,

$$L(T_i) = \frac{1}{\delta} \left( \frac{1}{B(T_i, T_{i+1})} - 1 \right), \quad i = 1, 2, \dots, M. \quad (4.1.21)$$

Also from equation (4.1.14), we have the following as LIBOR interest rate using the instantaneous forward rate:

$$L(T_i) = \frac{1}{\delta} \exp \left( - \int_{T_i}^{T_{i+1}} f(T_i, u) du \right), \quad i = 1, 2, \dots, M. \quad (4.1.22)$$

**4.1.9 Forward LIBOR Rate.** Forward LIBOR is the LIBOR negotiated at time  $t$ , comes to effect at time  $T_1$  and lasts for the period  $T_1 - T_2$  where the tenor  $\delta = T_2 - T_1$ . Here the rate last only for the interval  $[T_1, T_2]$  after which it ceases to exist. We consider the price of the bond  $B(t, T_2)$  for the above contract discounted with  $B(t, T_2)$ . The question is how do we price a bond with the above conditions to yield a maturity values of 1 unit of a currency. We do the same construction as shown in equation (4.1.21) to get the following

$$1 = (1 + \delta L(t, T_1, T_2)) \frac{B(t, T_2)}{B(t, T_1)}$$

hence *forward LIBOR rate* for the interval  $[T_1 - T_2]$  is given as

$$L(t, T_1, T_2) = \frac{1}{\delta} \left( \frac{B(t, T_1)}{B(t, T_2)} - 1 \right) \quad (4.1.23)$$

**4.1.10 Forward LIBOR Rate Processes (LIBOR with Tenor Dates).** Following the same analogy for forward LIBOR rate, assuming we have tenor structure as  $0 \leq T_0 \leq T_1 \leq T_2 \leq T_3 \cdots \leq T_{M+1}$  and given  $M$  bonds in the market as  $B(t, T_1), B(t, T_2), \dots, B(t, T_{M+1})$ , where the interval is  $\delta = T_{i+1} - T_i$  is the constant time difference of the bonds also known as tenor. We can construct all the different forward LIBOR rate for the intervals  $\delta$ . From equation (4.1.23) The LIBOR for the tenor structure is the forward LIBOR rate that is contracted at time  $t$  for the period  $T_i, T_{i+1}$  for  $i = 1, 2, \dots, M$  is called the forward LIBOR processes

$$L_i(t) = L(t, T_i, T_{i+1}) = \frac{1}{\delta} \left( \frac{B(t, T_i)}{B(t, T_{i+1})} - 1 \right). \quad (4.1.24)$$

The relationship between forward LIBOR processes from equation (4.1.24) and the forward bond price as shown in equation (4.1.3) is given by

$$f_B(t, T_i, T_{i+1}) = 1 + \delta L_i(t), \quad \forall t \in [0, T_M] \quad (4.1.25)$$

## 5. The Theory of LIBOR Market Model

In this chapter we give the theory of the LIBOR Market Model. We first give a discussion to show that LIBOR is tradable in the Market in section 5.1 and show the relationship between LIBOR and the bonds. In section 5.2, we set-up the model parameters for the LIBOR Forward rate model. In section 5.3, we show the construction of LIBOR forward rate model given bonds and volatilities from the market, and also solve the driftless LIBOR model and generalize our solution for driftless LIBOR. The notion of terminal measure for LIBOR is introduced in section 5.4 to find a solution for the forward rate LIBOR using only one measure. Section 5.5 is the Euler approximation of LIBOR. Here we show both the Stochastic differential(SDE) approximation of  $L_i(t)$  and that of  $\log L_i(t)$ . We also show the path simulation of the numeraire bond prices using the LIBOR paths simulated. In section 5.6, we show the graphical representation of the paths for the Terminal LIBOR and the path just before the Terminal LIBOR. We show when we have different Brownian process and when we simulate with the same Brownian process. The Monte Carlo simulation of LIBOR is done in section 5.7, where we price the value of caps. The major sources of reference for this chapter are Pelsser (2000), Musiela and Rutkowski (2004).

### 5.1 Defining the Bonds in the Market

Here we cannot go to the market and buy LIBOR, hence they are not tradable, but bonds are tradable in the market. Hence we can conveniently use bonds as numeraire based martingales in the measures we choose to work for our calculations. We let

$$\tilde{L}_i(t) = \delta L_i(t) \quad (5.1.1)$$

hence from equation (4.1.24) we have the following relation:

$$\tilde{L}_i(t) = \left( \frac{B(t, T_i)}{B(t, T_{i+1})} - 1 \right) = \frac{B(t, T_i) - B(t, T_{i+1})}{B(t, T_{i+1})} \quad (5.1.2)$$

$$\tilde{L}_i(t)B(t, T_{i+1}) = B(t, T_i) - B(t, T_{i+1}) \quad (5.1.3)$$

From equation (5.1.2), we have expresses LIBOR in terms of the bonds, which are the tradable assets in the market. Hence we deduce that LIBOR is also tradable. Also from equation (5.1.3) we deduce that  $\tilde{L}_i(t)B(t, T_{i+1})$  is also tradable. We derive the following:

$$\frac{B(t, T_i)}{B(t, T_{i+1})} = \tilde{L}_i(t) + 1 \quad (5.1.4)$$

$$\frac{B(t, T_n)}{B(t, T_{i+1})} = \prod_{k=n}^i (1 + \tilde{L}_k(t)) \quad (n \leq i) \quad (5.1.5)$$

At maturity  $B(T, T_i) = 1$ , we have the following

$$B(T_n, T_{i+1}) = \prod_{k=n}^i \frac{1}{(1 + \tilde{L}_k(T_n))} \quad (n \leq i) \quad (5.1.6)$$

Equation (5.1.4), shows that, bond prices determine the LIBOR rates at all time. We also deduce from equation (5.1.5), that numeraire-based is determined by the LIBOR rates at all times. From equation (5.1.6), the actual bond prices is determined at the tenor dates by the LIBOR rates at all times.

From equation 5.1.4, we have the following:

$$\hat{D}(t, T_i) = \frac{B(t, T_i)}{B(t, T_{i+1})} = \tilde{L}_i(t) + 1, \quad i = 1, \dots, n \quad (5.1.7)$$

then

$$B(t, T_i) = (\hat{D}(t, T_i) \hat{D}(t, T_i) \hat{D}(t, T_i) \dots \hat{D}(t, T_n)) B(t, T_{n+1}) \quad (5.1.8)$$

from equations (5.1.7) and (5.1.8) we conclude that, given all LIBOR rates  $\tilde{L}_i(t)$  for all tenor dates up to  $T_n$  and given  $B(t, T_{n+1})$ , then for any time  $t$ , we can determine all the bonds that expire up to the time  $T_{n+1}$  are determined at all times.

## 5.2 Setting Up the Parameters for LIBOR Forward Rate Model

Now we do the construction of LIBOR forward rate. We therefore state a theorem that will motivate the construction of the LIBOR rates as solutions of SDE, martingales in their own measure and also consistent with the bond prices at time 0 and have specified volatilities. Hence the bonds can be defined in terms of LIBOR rates, given a terminal bond  $B(t, T_{M+1})$  whose behaviour does not affect the dynamical equations of the LIBOR rates. These bond are assumed to follow an arbitrage free market.

### 5.2.1 Theorem. *Given*

1.  $B(t, T_{M+1}) > 0, \mathbb{Q}^{M+1}, W^{M+1}$ , where  $B(t, T_{M+1}) > 0$  is otherwise arbitrary price and the other two terms are numeraire measure and Brownian motion associated with the bonds  $B(t, T_{M+1})$ .
2.  $B^m(0, T_i), i = 1, \dots, M$ , positive strictly in  $i$ , where  $m$  indicates that this is a bond price from the market.
3.  $\sigma_i, i = 1, \dots, M - 1$  are deterministic non-negative functions on  $[0, T^*]$ .

Then there exist LIBOR rates  $\tilde{L}_i(t)$  whose logarithm have volatilities  $\sigma_i$  and bonds  $B(t, T_i), i = 1, \dots, M - 1$ , with initial conditions  $B(t, T_i(0)) = B(t, T_i^m(0))$  which form an arbitrage free bond market in which the LIBOR rates are consistent with the bond prices. The numeraire which makes all bond numeraire based martingale is  $B(t, T_{M+1})$ .

**5.2.2 Remark.** The dynamical equations satisfied by each  $L(t)$  is obtained in terms of the terminal Brownian motion  $W^{M+1}(t)$  during the construction.

### 5.3 Construction of LIBOR Market Model

Here we now construct the bond market described by the theorem 5.2.1. The construction is done by backwards induction, starting by defining the LIBOR rates  $L_M(t)$  as stochastic differential equations (SDE) where  $Q^{M+1}$  and  $W^{M+1}$  are the terminal measure and terminal Brownian motion respectively. We also define the bond price  $B(t, T_M)$  in terms of  $L_M(t)$  and also make the LIBOR rates a martingale in its own measures. Next we define  $L_{M-1}(t)$ , by backwards induction where  $Q^M$ ,  $W^M$  and  $B(t, T_M)$  are the measure, Brownian motion and the bonds for that interval considered. Followed by the SDE  $L_{M-1}$  where  $Q^{M-1}$  is the measure,  $W^{M-1}$  the Brownian motion associated with the SDE, and  $B(t, T_{M-1})$  the corresponding bonds and also make the corresponding LIBOR rates martingale in its own measures. Doing this backward induction, we succeed in defining all LIBOR rates and bonds in the model.

Hence we can assume that, the bonds  $B(t, T_{i+1}), B(t, T_{i+2}) \dots B(t, T_{M+1})$  have been defined as well as the corresponding measures  $Q^{i+1}, Q^{i+2} \dots, Q^{M+1}$  associated with the bonds, and the Brownian motions  $W^{i+1}, W^{i+2}, \dots, W^{M+1}$  associated with the measures are also defined. We also assume then that all LIBOR rates  $\tilde{L}_{i+1} \dots, \tilde{L}_{M+1}$  are defined. Then the driftless forward LIBOR rates process described is the solution of the following SDE

$$d\tilde{L}_i(t) = \sigma_i(t)L_i(t)dW_t^{i+1}, \quad i = 1, 2, \dots, M \quad (5.3.1)$$

$$\tilde{L}(0) = \frac{B^m(0, T_i)}{B^m(0, T_{i+1})} \quad (5.3.2)$$

$\sigma_i$  are the volatilities and  $W^{i+1}$  is the Brownian motion under the equivalent martingale measure  $Q^{i+1}$ . The SDE's in this case are called LIBOR Market Models but specifically for the Forward LIBOR rate process  $L_i(t)$ .

Here we consider the Radon-Nikodym theorem as in proposition 3.3.1 to arrive at the following:

$$d\mathbb{Q}^i = \xi_i d\mathbb{Q}^{i+1} \quad (5.3.3)$$

from equation (5.3.3), we arrive at the following

$$\xi_i(t) = \frac{B(t, T_i)/B(t, T_i(0))}{B(t, T_{i+1})/B(t, T_{i+1}(0))} \quad (5.3.4)$$

$$= \frac{B(t, T_{i+1}(0))}{B(t, T_i(0))} \times \frac{B(t, T_i)}{B(t, T_{i+1})} \quad (5.3.5)$$

Since  $(1 + \tilde{L}_i(t)) = \frac{B(t, T_i)}{B(t, T_{i+1})}$  from equation (5.1.5) we have the following

$$\xi_i(t) = \frac{B(t, T_{i+1}(0))}{B(t, T_i(0))} (1 + \delta \tilde{L}_i(t)) \quad (5.3.6)$$

From equation (5.1.4), we have the following:

$$B(0, T_i) = (1 + \tilde{L}(0))B(0, T_{i+1}) = \frac{B^m(0, T_i)}{B^m(0, T_{i+1})} B(0, T_{i+1}) = B^m(0, T_i) \quad \forall i \quad (5.3.7)$$



Here we conclude from equation (5.3.7) that, the bond that we have defined have initial values consistent with the market. Finally from equation (5.3.2), we conclude that the LIBOR rate and the bond prices are consistent in the market.

Here we find the solution for the driftless forward rate process when  $i = 1$

$$dL_1(t) = \sigma_i(t)L_1(t)dW^2 \quad (5.3.8)$$

We have the following solution for equation (5.3.8) using Theorem 3.2.4:

$$F(t, L_1(t)) = \log(L_1(t)), \quad L_1(t) = e^{F(t, L_1(t))}, \quad \frac{\partial F}{\partial L_1} = \frac{1}{L_1(t)}, \quad \frac{\partial^2 F}{\partial L_1(t)^2} = -\frac{1}{L_1(t)^2}. \quad (5.3.9)$$

Using the Itô's formula from equation (3.2.2) we get

$$dF(t, L_1(t)) = \frac{1}{L_1(t)} (\sigma(t)L_1(t)dW_t^2) + \frac{1}{2}\sigma(t)^2L_1(t)^2 \left( -\frac{1}{L_1(t)^2} \right) dt, \quad (5.3.10)$$

$$= -\frac{1}{2}\sigma(t)^2dt + \sigma(t)dW_t^2, \quad (5.3.11)$$

hence

$$F(t, L_1(t)) = F(0, L_1(0)) - \frac{1}{2}\sigma(t)^2dt + \sigma(t)dW_s^2. \quad (5.3.12)$$

We find  $L_1(t)$  by substituting equation (5.3.12) into the second term of equation (5.3.9) to get

$$L_1(t) = \exp \left( F(0, L_1(0)) - \frac{1}{2}\sigma(t)^2dt + \sigma(t)dW_t^2 \right),$$

$$L_1(t) = L_1(0) \exp \left( -\frac{1}{2} \int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s) dW_s^2 \right), \quad (5.3.13)$$

Generally we have the following for  $i = 1, 2, \dots, M$

$$L_i(t) = L_i(0) \exp \left( -\frac{1}{2} \int_0^t \sigma(s)^2 ds + \int_0^t \sigma(s) dW_s^{i+1} \right) \quad (5.3.14)$$

**5.3.1 Note.** When we follow the method above, we can express all the Forward LIBOR rates in their different equivalent martingale measures, coupled with different Brownian motions and they all happen to be martingales as in equation (5.3.14). By simplification, we want to express all the Forward LIBOR rates in their terminal measure<sup>1</sup>  $Q^{M+1}$ , where all the Forward LIBOR rates are expressed in one Brownian motion  $W^{M+1}$  called the terminal Brownian motion.

## 5.4 The LIBOR Market Model Under the Terminal Measure

In this section, the objective is to express all LIBOR rates in terms of its terminal measure.

Here we consider the Randon-Nikodym theorem as in proposition 3.3.1 to arrive at the following:

$$d\mathbb{Q}^i = \xi_i d\mathbb{Q}^{i+1}$$

---

<sup>1</sup>This is equivalent measure when  $i = M$  as  $Q^{M+1}$

From equation (5.3.6) we have the expression for  $\xi$  so we arrive at the following:

$$\frac{\xi_i(t)}{(1 + \delta \tilde{L}_i(t))} = \frac{B(t, T_{i+1}(0))}{B(t, T_i(0))}. \quad (5.4.1)$$

we find the derivative of (5.4.1) to get the following:

$$d\xi_i(t) = \frac{B(t, T_{i+1}(0))}{B(t, T_i(0))} d\tilde{L}_i(t) \quad (5.4.2)$$

We substitute equations (5.4.1) and (5.3.1) to get the following:

$$d\xi_i(t) = \frac{\sigma_i \tilde{L}_i(t)}{1 + \delta \tilde{L}_i(t)} \xi_i(t) dW^{i+1}(t). \quad (5.4.3)$$

When we take

$$q_i(t) = \frac{\sigma_i(t) \tilde{L}_i(t)}{1 + \delta \tilde{L}_i(t)} \quad (5.4.4)$$

we get

$$d\xi_i(t) = q_i(t) \xi_i(t) dW^{i+1}(t). \quad (5.4.5)$$

As we change the measure from  $Q^i$  to  $Q^{i+1}$ , there will be another Brownian component as stated in Theorem 3.3.7. Reasons being that there is drift term  $q_i(t)$  which enters the process as a result of the change of measure as shown in equation (5.4.5). This term is then subtracted from the old Brownian motion  $dW^{i+1}$  to have the following:

$$dW^i = dW^{i+1} - q_i dt \quad (5.4.6)$$

From equation (5.4.6) we can have the following:

$$dW^{i+1} = dW^{i+2} - q_{i+1} dt \quad (5.4.7)$$

$$dW^{i+2} = dW^{i+3} - q_{i+2} dt \quad (5.4.8)$$

$$dW^{i+3} = dW^{i+4} - q_{i+3} dt \quad (5.4.9)$$

$$dW^{i+3} = dW^{i+5} - q_{i+4} dt \quad (5.4.10)$$

If we substitute equation (5.4.8), (5.4.9), (5.4.10) into (5.4.7), to get

$$dW^{i+1} = dW^{i+3} - (q_{i+1} + q_{i+2}) dt \quad (5.4.11)$$

$$= dW^{i+4} - (q_{i+1} + q_{i+2} + q_{i+3}) dt \quad (5.4.12)$$

$$= dW^{i+5} - (q_{i+1} + q_{i+2} + q_{i+3} + q_{i+4}) dt \quad (5.4.13)$$

Hence generally continuing this process we have,

$$dW^{i+1} = dW^{i+p} - \left( \sum_{k=i+1}^{i+p-1} q_k \right) dt$$

Let  $M + 1 = i + p$

$$dW^{i+1} = dW^{M+1} - \left( \sum_{k=i+1}^M q_k \right) dt \quad (5.4.14)$$

We substitute equation (5.4.14) into (5.3.1) to get the following:

$$\frac{d\tilde{L}_i(t)}{\tilde{L}_i(t)} = \sigma_i dW^{M+1} - \left( \sum_{k=i+1}^M q_k \right) \sigma_i dt = \sigma_i dW^{M+1} - \left( \sum_{k=i+1}^M \frac{\sigma_i(t)\sigma_k(t)\tilde{L}_k(t)}{1 + \delta_k \tilde{L}_k(t)} \right) dt, \quad (1 \leq i \leq M) \quad (5.4.15)$$

we use equation (5.1.1) to replace  $\tilde{L}$  with  $\delta L$  to obtain

$$dL_i(t) = \sigma_i(t)L_i(t)dW^{M+1} - \left( \sum_{k=i+1}^M \frac{\delta_k \sigma_k(t)L_k(t)}{1 + \delta_k L_k(t)} \right) \sigma_i(t)L_i(t)dt, \quad (1 \leq i \leq M) \quad (5.4.16)$$

Alternatively, we have the solution using Stochastic differential approach as

$$L_i(t) = L_i(t) \exp \left( \left( - \sum_{k=i+1}^M \left( \frac{\delta_k \sigma_k(t)L_k(t)}{1 + \delta_k L_k(t)} \right) \sigma_i(t)L_i(t) - \frac{1}{2} \sigma_i^2(t) \right) \times dt + \sigma_i(T_k)L_i(t)dW^{M+1} \right) \quad (5.4.17)$$

Equation (5.4.16) or (5.4.17) is the dynamical equation of the LIBOR rates expressed in one Brownian motion  $W^{M+1}$  called the terminal Brownian motion. We observe that the volatility component  $\sigma_i$  remain unchanged after the change of measure. For  $i = M$ , the dynamical equation mentioned is a martingale since the summation part of equation (refnbc2) or (5.4.16) is zero, but this is not true for all other LIBOR rates for  $i \leq M - 1$  since we have a drift component as

$$\mu_i = - \sum_{k=i+1}^M \frac{\delta_k \sigma_k(t)L_k(t)}{1 + \delta_k L_k(t)} \sigma_i(t)L_i(t) \quad (5.4.18)$$

$\mu_i$  is also dependent on  $L_i(t)$  and the main LMM equation in (5.4.16) is non-linear in  $L_i(t)$  which makes the drift component complex to work with. All this contribute to make it tedious to find analytic solution for the LMM, so we need to resort to numerics. In this case Monte Carlo methods is ideal to use.

## 5.5 Simulation of LIBOR

Here we use the following Euler approximation method as follows to simulate LIBOR path [Pelsser \(2000\)](#)

$$L_i(T_{n+1}) = L_i(T_n) - \left( \sum_{k=i+1}^M \frac{\delta_k(T_n)\sigma_k(T_n)L_k(T_n)}{1 + \delta_k L_k(T_n)} \right) \sigma_i L_i(T_n)(T_{n+1} - T_n) + \sigma_i L_i(T_n)(W^{M+1}(T_{n+1}) - W^{M+1}(T_n)), \quad (5.5.1)$$

Alternatively, using the Stochastic Differential approach (SDE) approach we have the following approximation:

$$L_i(T_{n+1}) = L_i(T_n) \exp \left( \left( - \sum_{k=i+1}^M \left( \frac{\sigma_k \delta_k(T_n)L_k(T_n)}{1 + \delta_k L_k(T_n)} \right) \sigma_i(T_n) - \frac{1}{2} \sigma_i^2(T_n) \right) \times \Delta T + \sigma_i(T_n)(W_{T_{n+1}}^{M+1} - W_{T_n}^{M+1}) \right) \quad (5.5.2)$$

for

$$1 \leq i \leq M \quad n = 0, 1, 2, \dots, M$$

We show the path of LIBOR within a time interval  $[T_0 - T_n]$ .

Table 5.1: Matrix of Dependencies for LIBOR

T	$T_0$	$T_1$	$T_2$	$T_3$	$T_4$	$\dots$	$T_M$
$dW^{M+1}$		$dW^{M+1}(T_1)$	$dW^{M+1}(T_2)$	$dW^{M+1}(T_3)$	$dW^{M+1}(T_4)$	$\dots$	$dW^{M+1}(T_M)$
$L_0(T_M)$	$L_0(T_0)$						
$L_1(T_M)$	$L_1(T_0)$	$L_1(T_1)$					
$L_2(T_M)$	$L_2(T_0)$	$L_2(T_1)$	$L_2(T_2)$				
$L_3(T_M)$	$L_3(T_0)$	$L_3(T_1)$	$L_3(T_2)$	$L_3(T_3)$			
$L_4(T_M)$	$L_4(T_0)$	$L_4(T_1)$	$L_4(T_2)$	$L_4(T_3)$	$L_4(T_4)$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	
$L_M(T_M)$	$L_M(T_0)$	$L_M(T_1)$	$L_M(T_2)$	$L_M(T_3)$	$L_M(T_4)$	$\dots$	$L_M(T_M)$

We can see in table 5.1, how the previous LIBOR rates are the main dependencies for the current rates. Hence in the LMM, generally all the LIBOR rates are dependent on other LIBOR rates. Which means we have to simulate the path and save them as we simulate the next.

Here we consider the simulation of LIBOR rate, when the initial prices are  $L(0) = 0.055$  and the time change  $dt$  is 0.5, with volatility  $\sigma = 0.25$  and time step  $\delta = 0.5$ . Note that  $\delta$ ,  $\sigma$ , and  $dt$  are kept constant.

Table 5.2: Simulation of LIBOR Rate

$T$	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.0$	$T_3 = 1.5$	$T_4 = 2.0$	$T_5 = 2.5$	$T_6 = 3.0$
$dW^7$		0.41190457	-1.5530153	0.18849432	0.70103317	1.3397448	0.350829246
$L_0(T_M)$	0.055						
$L_1(T_M)$	0.055	0.05976992					
$L_2(T_M)$	0.055	0.05981993	0.03979823				
$L_3(T_M)$	0.055	0.05986998	0.03986773	0.04106791			
$L_4(T_M)$	0.055	0.05992007	0.03993737	0.04116483	0.04822863		
$L_5(T_M)$	0.055	0.05997021	0.04000717	0.04126206	0.04837308	0.06652079	
$L_6(T_M)$	0.055	0.06002039	0.04007712	0.04135958	0.04851812	0.06676965	0.07176026

Now we have the LIBOR prices  $L_i(T_n)$  from table 5.2, we can now calculate the numeraire bonds using the following formula:

$$B(\tau, T_n) = \prod_{j=n}^M (1 + \delta_j L_j(T_j))^{-1} \quad (5.5.3)$$

Table 5.3: Simulation of Numeraire Bond Prices

$T$	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.0$	$T_3 = 1.5$	$T_4 = 2.0$	$T_5 = 2.5$	$T_6 = 3.0$
$dW^T$		0.41190457	-1.5530153	0.18849432	0.70103317	1.3397448	0.350829246
$B_0(T_M)$	0.97323601						
$B_1(T_M)$	0.94718833	0.97098224					
$B_2(T_M)$	0.92183779	0.94278361	0.98048913				
$B_3(T_M)$	0.89716573	0.91538167	0.96132619	0.9798792			
$B_4(T_M)$	0.873154	0.88875455	0.94250559	0.96011766	0.97645349		
$B_5(T_M)$	0.84978491	0.86288098	0.92402184	0.94070985	0.95339419	0.96781025	
$B_6(T_M)$	0.82704128	0.83774023	0.90586952	0.92165031	0.93081352	0.9365439	0.96536266

## 5.6 Graphs of The Path of LIBOR

Here we show the path simulation of LIBOR rates with the following parameters  $M = 100$ ,  $\sigma = 0.15$ ,  $\delta = 0.5$ ,  $L(0) = 5\%$ . Figure 9.6, shows the LIBOR rates just before the Terminal values, figure 5.2 the Terminal LIBOR rate all under different Brownian component. Figure 5.3 is When the Terminal and the Path of LIBOR and the LIBOR just before the Terminal rates are superimposed together with the same Brownian component.

In figure 9.6, we simulate the LIBOR rate just before the terminal LIBOR, here with a different Brownian motion as the Terminal LIBOR.

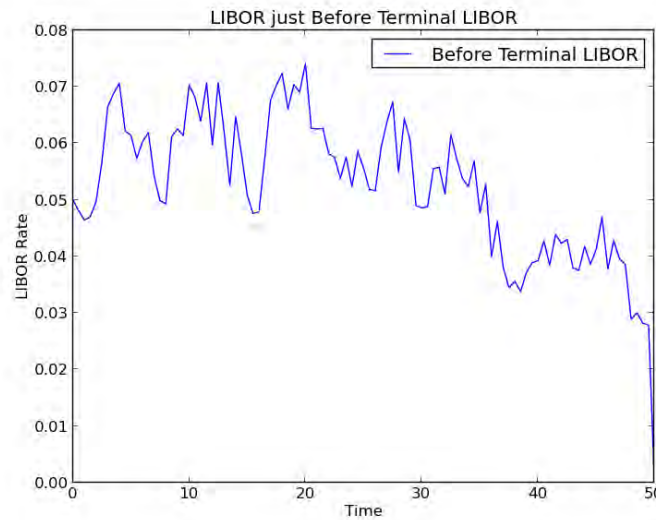
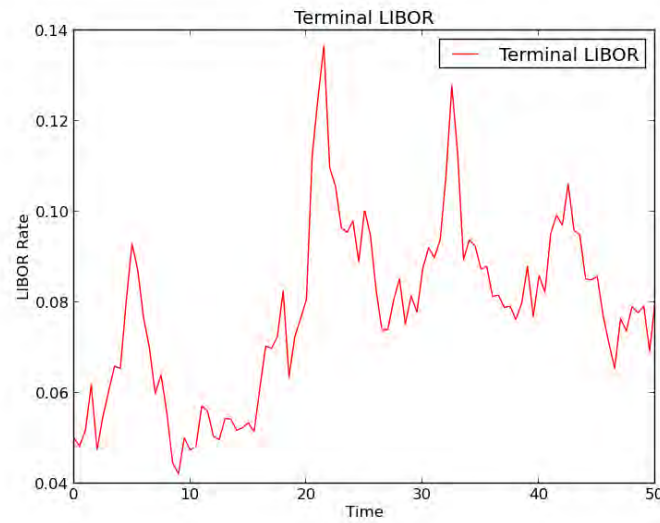
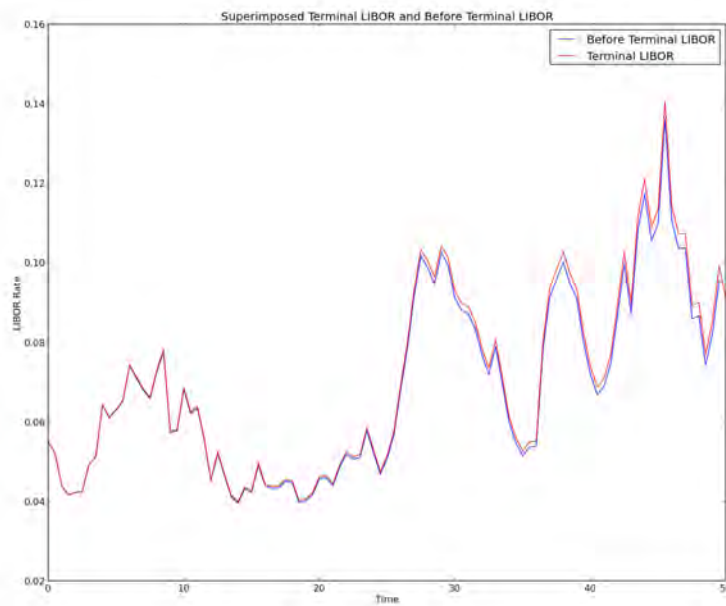


Figure 5.1: LIBOR Rate Path  $L_{M-1}$

The figure 5.2 the Terminal LIBOR with a different Brownian component.

Here we consider the same Brownian component. When we simulate the Terminal LIBOR and the LIBOR just before the terminal LIBOR, with the same Brownian path  $dW^{M+1}$ , they all have almost similar paths as shown in figure 5.3.

**5.6.1 Remark.** Here we conclude that when the LIBOR rates are simulated with the same Brownian component, we have almost the same path for the Terminal LIBOR  $L_M$  and the LIBOR path just before the terminal LIBOR  $L_{M-1}$  as shown in figure 5.3. Otherwise with a different Brownian path, the paths are different as shown in figures 9.6 and 5.2.

Figure 5.2: LIBOR Rate Path  $L_M$ Figure 5.3: Superimposed  $L_{M-1}$  and  $L_M$ 

## 5.7 Monte Carlo Simulation of LIBOR

“The main idea of the Monte Carlo method is to approximate an expected value  $E(X)$  by an arithmetic average of the results of a big number of independent experiments which all have the same distribution as  $X$ ” Korn et al. (2010b)

Two major theorems that are used in the implementation of Monte Carlo method are the strong

law of large numbers and the well known central limit theorem.

In probability theory, strong law of large numbers is a theorem that shows the relationship of performing the same experiment over and over again and the true value of the mean. The average of the number of experiments performed converges to the mean. Large trials has a higher tendency to approach the mean faster. This theorem forms the basis for Monte Carlo method. More details about Monte Carlo methods can be found in [Pelsser \(2000\)](#) and [Glasserman \(2005\)](#).

**5.7.1 Caps and Floors Pricing Using Monte Carlo Method.** A cap is a derivative that protects the holder from rises in interest rates for a given period. "A caplet is an insurance against high interest rate" [Pelsser \(2000\)](#). The way it works is the holder of the caplet is compensated when the underlying interest rate exceeds some pre-agreed limit by the seller of this instrument. Here the holder of the derivative is protecting the investment against rising of the underlying interest rate derivative. When a cap agreement is undertaken for the whole period of a given investment, example three years, reset at every three months, then at every three months there is a caplet payments if the conditions are satisfied. The sum of all the caplets for the individual accrual time is what constitute a cap. On the other hand floorlet is where the holder of the derivative is protecting the investment from a possible decline of the underlying interest rate derivative. Here a floor value is set, when the interest rate declines pass that value, the bearer of the floorlet is compensated. A single floor is called floorlet. Also the sum of all floorlet agreements that exist at every accrual period of the investment constitutes a floor. Here the pricing of cap is done in the framework of [Pelsser \(2000\)](#). Also more details about caps and floors can be found in [Brigo and Mercurio \(2006\)](#).

**5.7.2 Formula For Numeraire Based Cap Payoffs.** Here we have calculated the paths of LIBOR and the corresponding discount factors as bond prices, the the caplet payoff at time  $T_{i+1}$  is given by

$$V(T_{i+1}) = (L_i(T_i) - K)^+$$

where a single caplet is calculated as for a given strike value  $K$  and a single Brownian path considered for  $L_i(T_i)$ . Under the terminal measure  $Q^{M+1}$ , we also consider the terminal bond prices  $B^{M+1}$  as numeraire for the pricing of numeraire based cap. Here we use  $V^c$  to denote the payoff of the cap divided by the numeraire bond prices. The following formulations are can be used for the numereraire based bond prices.

$$V^c(T_{i+1}) = \frac{V(T_{i+1})}{B(t, T_{M+1})} \quad (5.7.1)$$

$$V^c(T_{i+1}) = V(T_{i+1}) \frac{B_{i+1}(T_{i+1})}{B(t, T_{M+1})} \quad (5.7.2)$$

We note that Equations (5.7.1) and (5.7.2) are equivalent from the relation below

$$V^c(T_i) = E^{M+1}(V^c(T_{i+1}) | \mathcal{F}_{T_i}) \quad (5.7.3)$$

**5.7.3 Estimated Numeraire Rebased Cap for  $N$  Caps.** Now we consider  $N$  paths of Brownian motion and calculate the corresponding caplets for every LIBOR path considered. Summing all these Caplets reduce to a Cap. Now we simulate  $N$  samples of such Cap and find the average to be the estimated Cap  $\hat{C}^{cap}$  as the Monte Carlo estimates for cap given as



$$\hat{C}^{cap} = \frac{1}{N} \sum_{l=1}^N C_l^{cap}$$

**5.7.4 Simulation of Numeraire Rebased Cap .** Here we consider the simulation of Cap rate, when the initial prices are  $L(0) = 0.05$ ,  $K = 0.5$  and the time change  $dt$  is 0.5, with volatility  $\sigma = 0.15$  and time step  $\delta = 0.5$ . Note that  $\delta$ ,  $\sigma$ , and  $dt$  are kept constant.

Table 5.4: Simulation of Cap Value

$T$	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.0$	$T_3 = 1.5$	$T_4 = 2.0$
$dW^5$		0.198512173223	0.204526632404	1.18115729765	-1.47486387064
$L_0(T_n)$	0.05				
$L_1(T_n)$	0.05	0.05118014			
$L_2(T_n)$	0.05	0.05119419	0.05246353		
$L_3(T_n)$	0.05	0.05120824	0.05249267	0.06229831	
$L_4(T_n)$	0.05	0.05122229	0.05252183	0.06235086	0.04969592
$B_5(T_n)$	0.88385429	0.90383035	0.92522027	0.94047218	0.97575449
$C_0(T_n)$	0.0				
$C_1(T_n)$	0.0	0.0	0.00127553		
$C_2(T_n)$	0.0	0.0	0.00129071	0.00261946	
$C_3(T_n)$	0.0	0.0	0.00130589	0.00265044	0.01260389
$C_4(T_n)$	0.0	0.0	0.00132108	0.00268145	0.01265775
Cap =					0.03840621

From table 5.4, the Cap rate is 3.84%, when LIBOR is capped at  $K = 5\%$ . The practical interpretation of a cap value of 3.8% is that the holder of the cap agreements enjoys 3.8% interest rate benefit.

Here we use the Monte Carlo simulation and draw  $N = 1000000$  samples to the estimates for cap as

$$\hat{C}^{cap} = \frac{1}{1000000} \sum_{i=1}^{1000000} C_i^{cap} = 0.02617314$$

where we get 2.6% rate.

We note that the approximate interest rate benefits is 2.6%, using the Monte Carlo method.

**5.7.5 Chapter Conclusion.** In this chapter, we have established the theory of LIBOR Market Model and considered the Monte Carlo Method for the pricing of caps and floors. It is realised that this approach is time consuming and not effective for more complicated derivative pricing. This stems from the complicated drift component of LMM. In the next chapters of this thesis, we adopt Lévy process as the underlining for the dynamic of LMM.

## 6. The Theory of Lévy Processes

Here we discuss and give some important definitions, theorems, lemmas, propositions and proofs on Lévy process that will be used in the main thesis. This chapter is structured into three sections. Section 6.1 is devoted to Poisson and Compound Poisson processes which examples of Lévy processes and will serve as building blocks for more general Lévy processes. Section 6.2 is devoted only to Levy processes. Then Section 6.3 is the discussion of more general Levy-Ito processes. Here we use the following books for the discussion of Lévy processes Applebaum (2004), Cont and Tankov (2004), Sato (1999), Øksendal and Sulem (2007).

### 6.1 Poisson Processes

**6.1.1 Definition. [Càdlàg Function Cont and Tankov (2004)]** A function  $f : [0, T] \rightarrow \mathbb{R}^d$  is said to be a càdlàg process if it is right-continuous with left limit for each  $t \in [0, T]$ .

**6.1.2 Definition. [Poisson Process Cont and Tankov (2004)]** Let  $(\tau_i)_{i \geq 1}$  be a sequence of independent exponential random variables with parameter  $\lambda$  and  $T_n = \sum_{i=1}^n \tau_i$ . The process  $(N_t, t \geq 0)$  defined by

$$N_t = \sum_{n \geq 1} 1_{t \geq T_n}$$

is called a Poisson process with intensity  $\lambda$ .

The Poisson process has the following properties

1. For any  $t > 0$ ,  $N_t$  is almost surely finite.
2. For any  $\omega$ , the sample path  $t \rightarrow N_t(\omega)$  is piecewise constant and increases by jumps of size 1.
3. The sample paths are Càdlàg.
4. For any  $t > 0$ ,  $N_{t-} = N_t$  with probability 1.
5.  $N_t$  is continuous in probability:

$$\forall t > 0, N_s \rightarrow N_t \quad \text{as} \quad s \rightarrow t$$

6. For any  $t > 0$ ,  $N_t$  follows a Poisson distribution with parameter  $\lambda t$ :

$$\forall n \in \mathbb{N}, P(N_t = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

7. The characteristic function of  $N_t$  is given by

$$E[e^{iuN_t}] = \exp\{\lambda t(e^{iu} - 1)\}, \forall u \in \mathbb{R}$$

8.  $N_t$  has independent increments: for any  $t_1 < \dots < t_n$ ,  $N_{t_n} - N_{t_{n-1}}, \dots, N_{t_2} - N_{t_1}, N_{t_1}$  are independent random variables.

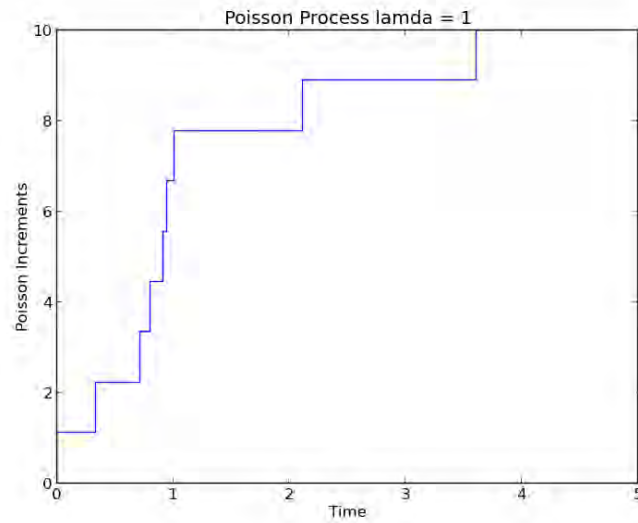


Figure 6.1: Non-Homogeneous Poisson Process

9. The increments of  $N$  are homogeneous: for any  $t > s$ ,  $N_t - N_s$  has the same distribution as  $N_{t-s}$ .

**6.1.3 Definition.** [Compound Poisson process [Cont and Tankov \(2004\)](#) (Page 70)] A compound Poisson process with intensity  $\lambda > 0$  and jump size distribution  $f$  is a stochastic process  $X_t$  defined as

$$X_t = \sum_{i=1}^{N_t} Y_i$$

where jump sizes  $Y_i$  are independently identically distributed with distribution  $f$  and  $N_t$  is a Poisson process with intensity  $\lambda$ , independent of  $(Y_i)_{i \geq 1}$ .

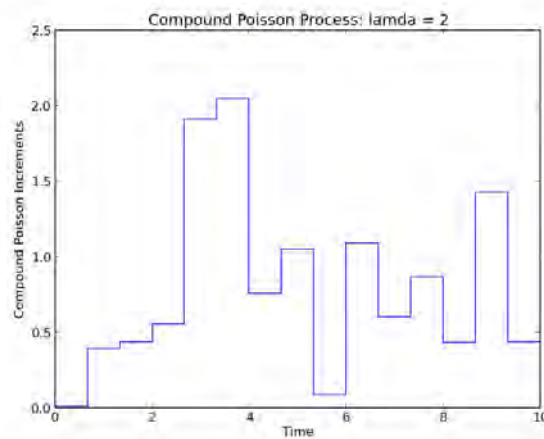


Figure 6.2: Compound Poisson Process

**6.1.4 Definition.** [Compensated Compound Poisson process **Papapantoleon (2005)**] Let  $N$  be a Poisson process with parameter  $\lambda$ . We shall call the process  $\tilde{N} = (\tilde{N}_t)_{0 \leq t \leq T}$  with  $\tilde{N}_t : \Omega \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$  where

$$\tilde{N}_t = N_t - \lambda t \quad (6.1.1)$$

a Compensated Poisson process.

**6.1.5 Proposition.** *The compensated Poisson process  $\tilde{N}_t$  is a martingale.*

*Proof.* Given

$$\begin{aligned} E[\tilde{N}_t \mid \tilde{N}_s] &= E(N_t - \lambda t \mid N_s), \\ &= E(N_t - N_s + N_s \mid N_s - N_0) - \lambda t, \quad \text{where } N_0 = 0 \\ &= E(N_t - N_s \mid N_s - N_0) + E(N_s \mid N_s) - \lambda t \\ &= \lambda(t - s) + N_s - \lambda t \\ &= N_s - \lambda s \\ &= \tilde{N}_s. \end{aligned} \quad (6.1.2)$$

□

Here we define Poisson random measure. Let  $\mathcal{B}_0$  be the family of Borel sets  $U \subset \mathbb{R}$  whose closure does not contain 0. For  $U \in \mathcal{B}_0$  we define

$$N(t, U) = N(t, U, \omega) = \sum_{s: 0 < s \leq t} \chi_U(\delta_{X_s}).$$

Here  $N(t, U)$  is called the number of jumps of size  $\delta_{X_s} \in U$  which occur before or at time  $t$ .  $N(t, U)$  is called the Poisson random measure of  $X_t$ .

**6.1.6 Theorem.** *Poisson random measure [Cont and Tankov (2004)]*

1. *The set function  $U \rightarrow N(t, U, \omega)$  defines a  $\sigma$ -finite measure on  $\mathcal{B}_0$  for each fixed  $t, \omega$ . The differential form of this measure is written  $N(t, dz)$ .*
- 2 *The set function  $[a, b) \times U \rightarrow N(b, U, \omega) - N(a, U, \omega); [a, b) \subset [0, \infty), U \in \mathcal{B}_0$  defines a  $\sigma$ -finite measure for each fixed  $\omega$ . The differential form of this measure is written as  $N(dt, dz)$ .*
- 3 *The set function*

$$\nu(U) = E[N(1, U)],$$

*where  $E = E_P$  denotes expectation with respect to  $P$ , also defines a  $\sigma$ -finite measure on  $\mathcal{B}_0$ , called the Lévy measure of  $X_t$ .*

- 4 *Fix  $U \in \mathcal{B}_0$ . Then the process*

$$\pi_U(t) := \pi_U(t, \omega) := N(t, U, \omega)$$

*is a Poisson process of intensity  $\lambda = \nu(U)$*

**6.1.7 Proposition.** [*Characteristic function of a compound Poisson process Cont and Tankov (2004)*] Let  $(X_{t \geq 0})$  be a compound Poisson process on  $\mathbb{R}^d$ . Its characteristic function has the following representation

$$E[\exp(i \cdot a \cdot X_t)] = \exp \left\{ t \lambda \int_{\mathbb{R}^d} (e^{i \cdot a \cdot x} - 1) f(dx) \right\}, \quad \forall a \in \mathbb{R}^d \quad (6.1.3)$$

where  $\lambda$  denote the jump intensity and  $f$  the jump size distribution.

*Proof.*

$$\begin{aligned} E[\exp(i \cdot \mu \cdot \sum_{i=1}^{N_t} Y_i)] &= \sum_{n \geq 0} \mathbb{E} \left[ \exp(i \cdot \mu \cdot \sum_{i=1}^{N_t} Y_i) \mid N = n \right] P(N = n) \\ &= \sum_{n \geq 0} \mathbb{E} \left[ \exp(i \cdot \mu \cdot \sum_{i=1}^{N_t} Y_i) \right] e^{-\lambda} \frac{\lambda^n}{n!} \end{aligned} \quad (6.1.4)$$

But

$$\begin{aligned} \mathbb{E} \left[ \exp(i \cdot \mu \cdot \sum_{i=1}^{N_t} Y_i) \right] &= \mathbb{E} (e^{iuY_1} e^{iuY_1} \dots e^{iuY_n}), \\ &= \mathbb{E}(e^{iuY_1}) \mathbb{E}(e^{iuY_1}) \dots \mathbb{E}(e^{iuY_n}), \\ &= \left( \int_{\mathbb{R}} e^{iux} f(dx) \right)^n, \end{aligned} \quad (6.1.5)$$

so we substitute equation (6.1.5) into (6.1.4) to get the following:

$$\begin{aligned} &= \sum_{n \geq 0} \left( \int_{\mathbb{R}} e^{iux} f(dx) \right)^n e^{-\lambda} \frac{\lambda^n}{n!} \\ &= e^{-\lambda} \sum_{n \geq 0} \frac{(\lambda \left( \int_{\mathbb{R}} e^{iux} f(dx) \right))^n}{n!} \\ &= e^{-\lambda} e^{\lambda \int_{\mathbb{R}} e^{iux} f(dx)} \\ &= e^{-\lambda \left( \int_{\mathbb{R}} e^{iux} f(dx) - 1 \right)} \end{aligned}$$

where  $\int_{\mathbb{R}} f(dx) = 1$  so we have the following

$$= \exp \left( \lambda \int_{\mathbb{R}} (e^{iux} - 1) f(dx) \right) \quad (6.1.6)$$

we introduce the measure  $\nu = \lambda f$ , equation (6.1.6) become

$$\mathbb{E} \left[ \exp(i \cdot \mu \cdot \sum_{i=1}^{N_t} Y_i) \right] = \exp \left( \lambda \int_{\mathbb{R}} (e^{iux} - 1) \nu(dx) \right) \quad (6.1.7)$$

□

## 6.2 Lévy Processes

**6.2.1 Definition.** [Lévy Process **Cont and Tankov (2004)**] A càdlàg stochastic process  $X_{t \geq 0}$  on  $(\Omega, \mathcal{F}, P)$  with values in  $\mathbb{R}^d$  is called a Lévy process if it possesses the following properties.

1.  $X_0 = 0$
2. Independent increments: for every increasing sequence of times  $t_0, \dots, t_n$ , the random variables  $X_{t_0}, X_{t_1} - X_{t_0}, \dots, X_{t_n} - X_{t_{n-1}}$  are independent.
3. Stationary increments: the law of  $X_{t+h} - X_t$  does not depend on  $t$ .
4. Stochastic continuity:  $\forall \varepsilon > 0, \lim_{h \rightarrow 0} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$ .
- 5 There is  $\omega_0 \in \mathcal{F}$  with probability  $P[\Omega_0] = 1$ , such that, for every  $\omega \in \omega_0$ ,  $X_t(\omega)$  is right continuous in  $t$ ,  $t \geq 0$  and has left limits in  $t > 0$ .

Here we refer to the definition of Brownian motion in 3.2.1 for the interpretations of condition (2) and (3).

The technical interpretations of (4), stochastic continuity is that given some the interval  $X_{t+h} - X_t$ , where  $h$  is a small change, there is a very very slim chance that it contains a jump .

A stochastic process that satisfies conditions (1), (2), (3), (4) is called a Lévy process in law.

**6.2.2 Theorem.** [Lévy Khinchin Representation **Cont and Tankov (2004)**] Let  $(X_t)_{t \geq 0}$  be a Lévy process on  $\mathbb{R}^d$  with the characteristic triplet  $(\mu, \sigma, \nu)$ . Where  $\mu \in \mathbb{R}$  is the drift, and  $\sigma \geq 0$  the volatility and  $\nu$  is called the Lévy measure.

Then

$$E[e^{iz \cdot X_t}] = \exp(t\psi), \quad z \in \mathbb{R}^d$$

with

$$\psi(z) = i\mu \cdot z - \frac{1}{2}z^2 \cdot \sigma^2 + \int_{\mathbb{R}^d} (e^{iz \cdot x} - 1 - iz \cdot x 1_{|x| \leq 1}) \nu(dx).$$

**6.2.3 Example.** Consider a Lévy process as follows:

$$X_t = \mu t + \sigma W_t + \sum_{i=1}^{N_t} Y_i - t\lambda k$$

where  $\mu \in \mathbb{R}$ ,  $\sigma \in \mathbb{R}_{\geq 0}$ ,  $(W_t)_{0 \leq t \leq T}$  is a standard Brownian motion,  $(N_t)_{0 \leq t \leq T}$  is a Poisson process with parameter  $\lambda$ , where  $E[N_t] = \lambda(t)$  and  $Y = (Y_i)_{i \geq 1}$  is an independently identically distributed sequence of random variables with probability distribution  $f$  and  $E[Y] = k < \infty$ .

$X_t$  is typical example of a Lévy process because, the Poisson process  $N_t$  is independent of the jump sizes  $Y_t$ . Also the the jump process  $N_t$  and  $Y_t$  are independent from the Brownian motion  $W(t)$ .

The characteristic function of  $X_t$  is given as

$$\mathbb{E}[e^{izX_t}] = \mathbb{E} \left[ \exp(iz(\mu t + \sigma W_t + \sum_{k=1}^{N_t} Y_i - t\lambda k)) \right]$$

All the sources or randomness are independent of each other so they can be separated as follows:

$$\begin{aligned}\mathbb{E}[e^{izX_t}] &= \exp(iz\mu t) \mathbb{E}[\exp(iz\sigma W(t)) \mathbb{E}[\exp(iz(\sum_{i=1}^{N_t} Y_i - t\lambda k))]] \\ &= \exp(iz\mu t) \mathbb{E}[\exp(iz\sigma W(t)) \mathbb{E}[\exp(iz(\sum_{i=1}^{N_t} Y_i - t\lambda k))]]\end{aligned}\quad (6.2.1)$$

we have

$$\mathbb{E}[\exp(iz\sigma W(t)) = e^{-\frac{1}{2}\sigma^2 z^2 t}, \quad W_t \sim N(0, t). \quad (6.2.2)$$

From equation (6.1.6) we get,

$$\mathbb{E}[\exp(iz(\sum_{i=1}^{N_t} Y_i))] = e^{\lambda t(\mathbb{E}[e^{izY} - 1])}, \quad N_t \sim \text{Poisson}(\lambda t) \quad (6.2.3)$$

We substitute (6.2.2) and (6.2.3) into (6.2.1) to get the following:

$$\begin{aligned}&= \exp(iz\mu t) \exp\left[-\frac{1}{2}\sigma^2 z^2 t\right] \exp[\lambda t(\mathbb{E}[e^{izY} - 1]) - \mathbb{E}(i\mu t\lambda k)] \\ &= \exp(iz\mu t) \exp\left[-\frac{1}{2}\sigma^2 z^2 t\right] \exp[\lambda t(\mathbb{E}[e^{izY} - 1 - i\mu Y])] \end{aligned}\quad (6.2.4)$$

The distribution of the jump process  $Y$  is  $f$ , hence substitute in equation (6.2.4) to get the following:

$$= \exp(iz\mu t) \exp\left[-\frac{1}{2}\sigma^2 z^2 t\right] \exp\left[\lambda t \left(\int_{\mathbb{R}} (e^{izx} - 1 - izx) f(dx)\right)\right]$$

Factorizing  $t$  from each component to get the following

$$\begin{aligned}&= \exp\left[t \left(i\mu z - \frac{1}{2}\sigma^2 z^2 \int_{\mathbb{R}} (e^{izx} - 1 - izx) \lambda f(dx)\right)\right] \\ &= \exp\left[t \left(i\mu z - \frac{1}{2}\sigma^2 z^2 \int_{\mathbb{R}} (e^{izx} - 1 - izx) \nu(dx)\right)\right]\end{aligned}\quad (6.2.5)$$

From equation (6.2.5), we can see that the product of the number of jumps  $\lambda$  and the distribution of jump size  $f$  is the Lévy measure  $\nu$  defined in theorem 6.2.2.

**6.2.4 Proposition.** [*Doléans-Dade Exponential Cont and Tankov (2004)*] Let  $(X)_{t \geq 0}$  be a Lévy process with Lévy triplet  $(\mu, \sigma, \nu)$ . Then there exist a unique cadlag process  $(Z)_{t \geq 0}$  such that

$$dZ_t = Z_{t-} dX_t, \quad Z_0 = 1,$$

We write  $Z = \mathcal{E}(X)$ , where  $Z$  is given by

$$Z_t = e^{X_t - \frac{\sigma^2 t}{2}} \prod_{0 \leq s \leq t} (1 + \Delta X_s) e^{-\Delta X_s}.$$

If  $\int_{-1}^1 |x| \nu(dx) < \infty$  then the jumps of  $X$  have finite variation and the stochastic exponential of  $X$  can be expressed as;

$$Z_t = e^{X_t - \frac{\sigma^2 t}{2}} \prod_{0 \leq s \leq t} (1 + \Delta X_s)$$

**6.2.5 Example.** Let

$$U = -\frac{1}{2} \int_0^t \gamma_u^2 du + \int_0^t \gamma_u dW_u \quad (6.2.6)$$

$$dU = -\frac{1}{2} \gamma_t^2 dt + \gamma_t dW_t \quad (6.2.7)$$

$$(dU)^2 = \left(-\frac{1}{2} \gamma_t^2 dt + \gamma_t dW_t\right) \left(-\frac{1}{2} \gamma_t^2 dt + \gamma_t dW_t\right) \quad (6.2.8)$$

$$= \gamma_t^2 dt \quad (6.2.9)$$

Let

$$Y_t = e^{\left(-\frac{1}{2} \int_0^t \gamma_u^2 du + \int_0^t \gamma_u dW_u\right)} = e^U$$

$$Y_t = e^U \quad (6.2.10)$$

Note that  $U$  and  $Y$  are continuous.

From the Itô formula we get

$$df(X_t) = \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2$$

$$dY = e^U du + \frac{1}{2} e^U (dU)^2$$

$$= e^U \left(-\frac{1}{2} \gamma_t^2 dt + \gamma_t dW_t\right) + \frac{1}{2} e^U (\gamma_t^2 dt)$$

$$dY = e^U \gamma_t dW_t \quad (6.2.11)$$

Let

$$X_t = \int_0^t \gamma_u dW_u$$

$$dX_t = \gamma_t dW_t \quad (6.2.12)$$

We substitute equation (6.2.12) and (6.2.10) into (6.2.11) to get the following:

$$dY = Y_t dX_t$$

so that

$$Y = \mathcal{E}(X_t).$$

From the proposition (6.2.4) we have

$$\mathcal{E}(\mu X) = \exp \left\{ \mu X_t - \frac{1}{2} \mu^2 t \right\}$$

**6.2.6 Theorem.** *[Itô-Lévy Decomposition Øksendal and Sulem (2007)] Let  $(X)_{t \geq 0}$  be a Lévy process. Then  $(X)_{t \geq 0}$  has the decomposition*

$$(X)_{t \geq 0} = \alpha t + \sigma W(t) + \int_{|z| < R} z \tilde{N}(t, dz) + \int_{|z| \geq R} z N(t, dz),$$

for some constants  $\alpha \in \mathbb{R}, \sigma \in \mathbb{R}, R \in [0, \infty]$ . Where

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt$$



$\tilde{N}(dt, dz)$  is the compensated Poisson random measure of  $(X)_{t \geq 0}$ , and  $W(t)$  is a Brownian motion independent of  $\tilde{N}(dt, dz)$ . For each  $A \in W_0$  the process

$$M_t := \tilde{N}(t, A) \text{ is a martingale}$$

If  $\alpha = 0$  and  $R = \infty$ , then  $(X)_{t \geq 0}$  is a martingale which we call a Lévy martingale.

Here we wish to state a simpler version of the Girsanov theorem for jump processes as stated in Øksendal (2007) as follows:

**6.2.7 Theorem.** *[(Girsanov theorem I for jump processes)] Let  $X(t)$  be an 1-dimensional Itô process of the form*

$$dX(t) = \alpha(t, \omega)dt + \int_R \gamma \tilde{N}(dt, dz).$$

Assume that there exists a process  $\theta(t) \leq 1$  such that

$$\gamma(t, z)\theta(t, z)\nu(dz) = \alpha(t), \text{quad for almost all } (t, \omega)$$

and such that the process  $Z(t)$  defined by

$$Z(t) = \exp \left\{ \int_0^t \int_R \ln(1 - \theta(s, z)) \tilde{N}(ds, dz) + \int_0^t \int_R \{\ln(1 - \theta(s, z)) + \theta(s, z)\} \nu(ds) dz \right\}$$

exists for  $0 \leq t \leq T$ . Define a measure  $Q$  on  $\mathcal{F}_T$  by

$$dQ(\omega) = Z(T)dP(\omega)$$

Assume that

$$E_P[Z(T)] = 1.$$

Then  $Q$  is equivalent local martingale measure for  $X(t)$ .

**6.2.8 Proposition.** *[Exponential of a Lévy Process Cont and Tankov (2004)(page 284)] Let  $(X_t)_{t \geq 0}$  be a Lévy process with Lévy triplet  $(\sigma^2, \nu, \gamma)$  satisfying*

$$\int_{|y| \leq 1} e^y \nu(dy) < \infty$$

Then  $Y_t = \exp(X_t)$  is a semi-martingale with decomposition  $Y_t = M_t + A_t$  where the martingale part is given by

$$M_t = 1 + \int_0^t Y_{s-} \sigma dW_s + \int_{[0, t] \times \mathbb{R}} Y_{s-} (e^z - 1) \tilde{N}_X(dt, dz)$$

and the continuous finite variation drift is given by

$$A_t = \int_0^t Y_{s-} \left[ \gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z1_{|z| \leq 1}) \nu dz \right] ds$$

$(Y_t)$  is a martingale if and only if

$$\gamma + \frac{\sigma^2}{2} + \int_{-\infty}^{\infty} (e^z - 1 - z1_{|z| \leq 1}) \nu dz = 0$$

**6.2.9 Theorem.** [*The One-Dimensional Itô Formula Øksendal and Sulem (2007)*] Suppose  $X(t) \in \mathbb{R}$  is an Itô-Lévy process of the form

$$dX(t) = \alpha(t, \omega)dt + \beta(t, \omega)dW(t) + \int_{\mathbb{R}} \gamma(t, z, \omega) \tilde{N}(dt, dz)$$

where

$$\tilde{N}(dt, dz) = \begin{cases} N(dt, dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt, dz) & \text{if } |z| \geq R \end{cases}$$

for some  $R \in [0, \infty]$ . Let  $f \in C^2(\mathbb{R}^2)$  and define  $Y(t) = f(t, X(t))$ . Then  $Y(t)$  is again an Itô-Lévy process and

$$\begin{aligned} dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t, \omega)dt + \beta(t, \omega)dB(t)] \\ &\quad + \frac{1}{2}\beta^2(t, \omega)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt \\ &\quad + \int_{|z| < R} \{f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-)) - \frac{\partial f}{\partial x}(t, X(t^-))\gamma\}\nu(dz)dt \\ &\quad + \int_{\mathbb{R}} \{f(t, X(t^-) + \gamma(t, z)) - f(t, X(t^-))\}\tilde{N}(dt, dz) \end{aligned} \quad (6.2.13)$$

**6.2.10 Example.** Suppose that

$$dX(t) = \alpha dt + \sigma dW(t) + \int_{\mathbb{R}} \gamma(z) \tilde{N}(dt, dz), \quad X(0) = x \in \mathbb{R}$$

where  $\alpha, \sigma$  are constants,  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  is a given function. We use Itô's formula for Lévy process to find  $dY(t)$  when

$$Y(t) = \exp(X(t))$$

**6.2.11 Solution.** Let

$$f(t, X(t)) = Y(t) = \exp(X(t)) \quad (6.2.14)$$

We use theorem 6.2.9 for  $f(t, X(t))$  to get the following

$$dY(t) = \exp(X(t))[\alpha dt + \sigma dW(t)] + \frac{1}{2}\sigma^2 \exp(X(t))dt \quad (6.2.15)$$

$$\begin{aligned} &+ \int_{z < R} \{\exp(X(t^-) + \gamma(z)) - \exp(X(t^-)) - \gamma(z) \exp(X(t^-))\} \nu(dz)dt \\ &+ \int_{z > R} \{\exp(X(t^-) + \gamma(z)) - \exp(X(t^-))\} \tilde{N}(dt, dz) \end{aligned} \quad (6.2.16)$$

$$dY(t) = \exp(X(t)) \left[ [\alpha dt + \sigma dW(t)] + \frac{1}{2}\sigma^2 dt + \int_{z < R} \{e^{\gamma(z)} - 1 - \gamma(z)\} \nu(dz)dt + \int_{z > R} \{e^{\gamma(z)} - 1\} \tilde{N}(dt, dz) \right] \quad (6.2.17)$$

We substitute equation (6.2.14) into (6.2.17), and group terms to get the following:

$$dY(t) = Y(t) \left[ \left\{ \alpha + \frac{1}{2}\sigma^2 + \int_{z < R} \{e^{\gamma(z)} - 1 - \gamma(z)\} \nu(dz) \right\} dt + \left\{ \sigma dW(t) + \int_{z > R} \{e^{\gamma(z)} - 1\} \tilde{N}(dt, dz) \right\} \right] \quad (6.2.18)$$

**6.2.12 Change of Measure.** Let

$$dH_t = h dW_t + \int_{\mathbb{R}} \theta(s, z) \tilde{N}(dt, dz) \quad (6.2.19)$$

and let

$$G_t = \mathcal{E}(H) \quad (6.2.20)$$

by definition we have

$$\frac{dG_t}{G_t} = h dW_t + \int_{\mathbb{R}} \theta(s, z) \tilde{N}(dt, dz) R \quad (6.2.21)$$

Note that  $G_t$  is a martingale since it is multiplied by the sum of two martingales.

**6.2.13 Theorem.** Let  $\xi(t, z)$  be arbitrary with sufficient smoothness. Let

$$W_t^h = W_t - \int_0^t h_s ds, \quad (6.2.22)$$

$$dW_t^h = dW_t - h_t dt, \quad (6.2.23)$$

and let

$$dJ = \int_{\mathbb{R}} \xi(t, z) N(dt, dz) - (1 + \theta(s, z)) d\nu(z) dt, \quad (6.2.24)$$

$$= \xi(t, z) (dt, dz) - \int_{\mathbb{R}} \xi(t, z) (1 + \theta(s, z)) d\nu(z) dt, \quad (6.2.25)$$

$$= \int_{\mathbb{R}} \xi(t, z) N(dt, dz) - (\theta(s, z)) d\nu(z) dt \quad (6.2.26)$$

then both  $W^h$  and  $J$  are local martingales with respect to the changed measure  $dQ = Z dP$ , where  $P$  is the original measure, and  $W^h$  is a Brownian motion.

**6.2.14 Theorem.** Let

$$dX_t = b_t + c^{1/2} dW_t + \int_{\mathbb{R}} \gamma(t, z) N(dt, dz) \quad (6.2.27)$$

assume that we are given  $\theta, h(t)$  such that  $G$  is given by

$$G(t) = \mathcal{E} \left( \int_0^t h dW_s + \int_{\mathbb{R}} \theta \tilde{N}(dt, dz) \right)$$

then we have

$$dX = b_t + c^{1/2} (dW_t - h dt) + c^{1/2} h dt \int_{\mathbb{R}} \gamma(t, z) N(dt, dz) - \theta d\nu(z) dt + \int_{\mathbb{R}} \gamma \theta d\nu(z) dt, \quad (6.2.28)$$

$$= \left( b_t + c^{1/2} h + \int_{\mathbb{R}} \gamma \theta d\nu(z) dt \right) dt + c^{1/2} dW_t + \int_{\mathbb{R}} \gamma(t, z) \left( \tilde{N}(dt, dz) - \theta d\nu(z) dt \right), \quad (6.2.29)$$

$$= \left( b_t + c^{1/2} h + \int_{\mathbb{R}} \gamma \theta d\nu(z) dt \right) dt + c^{1/2} dW_t + \int_{\mathbb{R}} \gamma(t, z) (N(dt, dz) - (1 + \theta) d\nu(z) dt), \quad (6.2.30)$$

hence  $X_t$  is a (local)  $P$  martingale if and only if

$$b + c^{1/2}h + \int_R \gamma \theta d\nu(z)dt = 0. \quad (6.2.31)$$

Further under the new measure,  $X_t$  is a generalised Lévy process with drift term  $b + c^{1/2}h + \int_R \gamma \theta d\nu(z)dt$ , volatility  $c^{1/2}$  and Lévy measure  $(1 + \theta)d\nu(z)$ . The Brownian motion is given by  $dW_t^h = dW_t - hdt$

## 6.3 More General Lévy Itô Processes

**6.3.1 Homogeneous and Non-Homogeneous Lévy process.** The two major kinds of Lévy processes are Homogeneous and non-homogeneous processes. When the parameters of the Lévy process are not dependent on time ( $t$ ), they are called homogeneous Lévy processes, otherwise they are said to be non-homogeneous. This happens when the stationarity assumption is relaxed. This comes with more flexibility with the Lévy process. In our study we will only consider non-homogeneous Lévy processes because they are more flexible, and general process in working with than the homogeneous counterpart, because the corresponding parameters are constants as compared to time dependent in the non-homogeneous case.

**6.3.2 Definition.** A time-inhomogeneous Lévy process, is an adapted, càdlàg  $\mathbb{R}^d$ -valued stochastic process  $L = (L_t)_{0 \leq t \leq T}$  with  $L_0 = 0$ , such that the following conditions are true.

- D1.  $L$  has independent increments.
- D2. The law of  $L_t$ , for all  $t \in [0, T]$ , is described by the characteristics function.

$$E[e^{u, L_t}] = \exp \int_0^t \left( i \langle u, b_s \rangle - \frac{1}{2} \langle u, c_s u \rangle + \int_{\mathbb{R}^d} (e^{i \langle u, z \rangle} - 1 - i \langle u, h_z \rangle) \nu(ds, dz) \right) ds$$

where  $b_t \in \mathbb{R}^d$ ,  $c_t$  is a symmetric definite  $d \times d$  matrix and  $\nu_s$  is a Lévy measure on  $\mathbb{R}^d$ , that it satisfies  $\nu_t(\{0\}) = 0$  and  $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu_t(dx) < \infty$  for all  $t \in [0, T]$ . Moreover, Assumptions (AC) hold. Assumption (AC). The triplets  $(b_t, c_t, \nu_t)$  satisfy

$$\int_0^T \left( |b_t| + |c_t| + \int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu_s \right) dt < \infty.$$

Assumption (EM). There exist a constant  $M > 1$ , such that the Lévy measure  $\nu$  satisfy

$$\int_0^T \exp(\langle u, z \rangle) \nu(dz) dt < \infty, \forall u \in [-M, M]^d.$$

Moreover, without loss of generality, we assume  $\int_D \exp(\langle u, z \rangle) \nu_t(dz) < \infty$  for all  $t \in [0, T]$  and  $u \in [-M, M]^d$ .

An example of time inhomogeneous Lévy process is given as

$$dL_t^{T*} = b_t dt + c_t^{\frac{1}{2}} dW_t^{T*} + \int_R z \tilde{N}_1(dt, dz)$$

Here we discuss the notions of Characteristic function, moment generating function and the cumulant generating function. Characteristic function of a random variable is known as the Fourier transform of its distribution. The characteristic function always exists. The moment generating function is not always defined and may not exist at all. The log of the moment generating function gives the cumulant generating function. We give the formal definitions as in [Cont and Tankov \(2004\)](#).

**6.3.3 Definition. Characteristic Function** The characteristic function of  $\mathbb{R}$ -valued random variable  $X$  is the function  $\Phi_X : \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$\forall z \in \mathbb{R}, \Phi_X(z) = E[\exp(iz \cdot X)] = \int_{\mathbb{R}} e^{iz \cdot x} d\mu_X(x).$$

**6.3.4 Definition. Moment Generating Function** The moment generating function of  $\mathbb{R}$ -valued random variable  $X$  is the function  $M_X$  defined by

$$\forall u \in \mathbb{R}, M_X(u) = E[\exp(u \cdot X)].$$

**6.3.5 Remark.** We note that the moment generating function, does not always exist, but when it does, it is related to the characteristic function as

$$M_X = \Phi(-iu)$$

**6.3.6 Definition. Cumulant Generating Function** Let  $X$  be a random variable and  $\Phi_X$  its characteristic function, where  $\Phi_X(0) = 1$  and  $\Phi_X$  is continuous at  $z = 0$ , so  $\Phi_X(z) \neq 0$  in the neighbourhood of  $z = 0$ . One can then define continuous version of the logarithm of  $\Phi_X$ : there exists a unique continuous function  $\Psi$  defined in the neighbourhood of zero such that:

$$\Psi_X(0) = 0, \quad \text{and} \quad \Psi_X(z) = \log(\Phi_X(z))$$

the function  $\Psi$  is called the log-characteristic function or the cumulant generating function

**6.3.7 Normal Inverse Gaussian Distribution (NIG)** [[Barndorff-Nielsen et al. \(2001\)](#), [Hakwa \(2001\)](#)]. NIG process is a time-changed Brownian motion subordinated<sup>1</sup> by inverse Gaussian process. Details about subordination can be found in [Asmussen and Glynn \(2007\)](#). The way we model this process is to time-change standard Brownian motion  $\{W_t, t \geq 0\}$ , where the drift is given by inverse Gaussian process  $\{I_t, t \geq 0\}$ , with parameters  $a = 1$  and  $b = \sqrt{\alpha^2 - \beta^2}$ . The resulting stochastic process for the NIG process is given by

$$X_t = \beta \delta^2 I_t + \delta W_{I_t} \tag{6.3.1}$$

with parameters  $\alpha, \beta$  and  $\delta$ . The algorithm and python code for simulating the NIG process is seen in [appendix C](#). The density of NIG distribution is given by

$$NIG(x; \alpha, \beta, \mu, \delta) = a(\alpha, \beta, \mu, \delta) q\left(\frac{x - \mu}{\delta}\right)^{-1} K_1\left\{\delta \alpha q\left(\frac{x - \mu}{\delta}\right)\right\} e^{\beta x} \tag{6.3.2}$$

where

$$a(\alpha, \beta, \mu, \delta) = \pi^{-1} \delta \exp\{\delta \sqrt{\alpha^2 - \beta^2} - \beta \mu\}, \tag{6.3.3}$$

---

<sup>1</sup>Subordinators are non-decreasing Lévy processes: examples of subordinators are Poisson process, gamma process, increasing compound Poisson process

$$q(x) = \sqrt{1 + x^2}, \quad (6.3.4)$$

$K_1$  is the modified bessel function of the third kind.  $\mu \in \mathbb{R}$ ,  $\delta \in \mathbb{R}_+$  and  $0 \leq \beta < \alpha$ . The characteristic function of NIG process is given by

$$C(u; \alpha, \beta, \mu, \delta) = \exp \left\{ \delta \left\{ \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \mu)^2} \right\} + u\mu \right\}. \quad (6.3.5)$$

We find the log of equation (6.3.5), to get the cumulant generating function  $K$ , for the NIG process as

$$K(u; \alpha, \beta, \mu, \delta) = \delta \left\{ \sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + \mu)^2} \right\} + u\mu. \quad (6.3.6)$$

To simplify equation (6.3.2), we write the following from equation (6.3.4):

$$q\left(\frac{x - \mu}{\delta}\right) = \frac{\sqrt{\delta^2 + (x - \mu)^2}}{\delta}, \quad (6.3.7)$$

$$q\left(\frac{x - \mu}{\delta}\right)^{-1} = \frac{\delta}{\sqrt{\delta^2 + (x - \mu)^2}}, \quad (6.3.8)$$

we substitute equations (6.3.7), (6.3.8) and (6.3.3) into (6.3.2) to get the following as the density of NIG process:

$$NIG(x; \alpha, \beta, \mu, \delta) = \frac{\alpha\delta}{\pi} \frac{\exp\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\}}{\sqrt{\delta^2 + (x - \mu)^2}} K_1 \left\{ \alpha\sqrt{\delta^2 + (x - \mu)^2} \right\}. \quad (6.3.9)$$

Also the Lévy measure of the NIG distribution is given by

$$\nu_{NIG}(\alpha, \beta, \delta)(dx) = \frac{\delta\alpha}{\pi|x|} \exp(\beta x) K_1(\alpha|x|) dx. \quad (6.3.10)$$

## 7. The Lévy-LIBOR Model

In this chapter we introduce the notion of Lévy LIBOR model, by this we model the dynamic of LIBOR using Lévy framework as elaborated in chapter 6. “The Lévy LIBOR model is a market model where the forward LIBOR rate is modelled directly and is driven by time inhomogeneous Lévy process” as Papapantoleon and Skovmand (2010) put it. The theory and notations used here is in the framework of Eberlein and Özkan (2005). In section 7.1, we give the parameters and initial conditions needed for the construction of the Lévy LIBOR Model. In section 7.2 we start the construction of Lévy LIBOR for the interval  $[T_1^*, T^*]$  by backwards induction using theorem 7.1.2. We follow the same procedure as in section refSecondconstruction, but with a new time interval  $[T_2^*, T_1^*]$ . In Section 7.4, we give a generalisation for the construction of Lévy LIBOR model. Finally in section 7.5, we give the dynamics of Lévy LIBOR process under the terminal measure.

### 7.1 Lévy-LIBOR Setting

Here the terminal time-inhomogeneous Lévy process considered is given as  $L_t^{T^*}$ :

$$dL_t^{T^*} = b_t dt + c_t^{\frac{1}{2}} dW_t^{T^*} + \int_R z \tilde{N}_1(dt, dz) \quad (7.1.1)$$

where  $W, c^{1/2}$ , are the Brownian motion and the volatility of the Brownian motion.  $\tilde{N}_1 = \mu^L - \nu^{T^*, L}$  is the compensator where  $\mu^L$  and  $\nu^{T^*, L}$  are the random measure associated with the jumps, Lévy measure of the Lévy process  $L_t^{T^*}$  respectively. The drift  $b_t$  is specified later in equations (7.2.2).

**7.1.1 Assumptions of Given Lévy Process  $L_t^{T^*}$ .** In this section, we state all the assumptions that are underlying the given Lévy process  $L_t^{T^*}$  in equation (7.1.1). These assumptions makes the process defined and finite.

We assume the following integral is finite from equation (7.1.1)

$$\int_0^{T^*} (|b_s| + c_s) ds < \infty. \quad (7.1.2)$$

We also assume that the Lévy measures  $\nu$  is a time inhomogeneous and use the notation  $\nu = \nu_s$ , which are also measures on  $\mathbb{R}$  with  $\nu(\{0\}) = 0$  is also finite for the following

$$\int_0^{T^*} \int_{\mathbb{R}} (z^2 \wedge 1) \nu(dz) ds < \infty \quad (7.1.3)$$

and also satisfy the following additional integrability assumption as

$$\int_0^{T^*} \int_{\{|z|>1\}} \exp(ux) \nu(dz) ds < \infty \quad (7.1.4)$$

for some  $u \leq (1 + \epsilon)M$  where  $M, \epsilon > 0$  are constants such that  $\sum_{i=1}^n |\lambda(\cdot, T_i^*)| \leq M$ .

Now we have defined the Lévy process for the construction of Lévy LIBOR model, we also give a theorem to set all the parameters needed for the construction of the Lévy LIBOR model and also define the assumptions underlying the model.

**7.1.2 Theorem.** *Given:*

1.  $B(t, T_{M+1}) > 0, \mathbb{Q}^{M+1}, W^{M+1}$ , where  $B(t, T_{M+1}) > 0$  is otherwise arbitrary price and the other two terms are numeraire measure and Brownian motion associated with the bonds  $B(t, T_{M+1})$
2.  $B^m(0, T_i), i = 1, \dots, M$ , positive strictly in  $i$ , where  $m$  indicates that this is a bond price from the market.
3.  $\sigma_i, i = 1, \dots, M - 1$  are deterministic non-negative functions on  $[0, T^*]$ .
4. For any maturity  $T_i$ , we are given a  $R^1$ -valued, bounded  $\mathbb{F}$ -adapted process  $\lambda(t, T_i)$  which are deterministic non-negative, representing the volatility of the forward LIBOR process  $L(t, T_i)$ . Moreover

$$\sum_{i=1}^n |\lambda(s, T_i)| < M, \quad \forall s \in [0, T^*] \quad (7.1.5)$$

where  $M$  is a constant from assumption (EM), and  $\lambda(s, T_i) = 0$  for all  $s > T_i$ .

Then there exist LIBOR rates  $\tilde{L}_i(t)$  whose logarithms have volatilities  $\lambda_i$  and bonds  $B(t, T_i), i = 1, \dots, M - 1$ , with initial conditions  $B(t, T_i(0)) = B(t, T_i^m(0))$  which form an arbitrage free bond market in which the LIBOR rates are consistent with the bond prices. The numeraire which makes all bond numeraire based martingale is  $B(t, T^*)$ .

Next we construct the bond market indicated in theorem (7.1.2). Note that this construction is done by backwards induction:

- We first define  $L(t, T_1^*)$  as the solution of an SDE using  $L_t^{T^*}$  (defined in terms of measure  $P_{T^*}, W^{T^*}, \mu^L, \nu^{T^*}$ ). Here We ensure that the process is a martingale in it own measure.
- Next we define the bond  $B(t, T_1^*)$  using  $L(t, T_1^*)$  and  $B(t, T^*)$ .
- Define the  $P_{T_1^*}$  as the forward measure from  $T^*$  with respect to the numeraire  $B(t, T_1^*)$  and the Brownian motion  $W^{T^*}$ . Also  $\nu^{T_1^*}$  is defined as the compensator. We use the information above to get the definition of  $L_{T^*}$
- We repeat the previous three steps by defining  $L(t, T_2^*)$  as the solution of an SDE using  $L_t^{T_1^*}$  (defined in terms of measure  $P_{T_1^*}, W^{T_1^*}, \mu^L, \nu^{T_1^*}$ ). Here We ensure that the process is a martingale in it own measure.
- Next we define the bond  $B_{t, T_2^*}$  using  $L(t, T_2^*)$  and  $B(t, T_1^*)$ .
- Define the  $P_{T_2^*}$  as the forward measure from  $T_1^*$  with respect to the numeraire  $B(t, T_1^*)$  and the Brownian motion  $W^{T_1^*}$ . Also  $\nu^{T_2^*}$  is defined as the compensator. We use the information above to get the definition of  $L_{T_2^*}$
- When we repeat this process many times we end up defining all the bonds and LIBOR rates in the model.



**7.1.3 Time Interval for Backwards Induction.** Since we work by backwards induction, the tenor interval will be given as  $T^*, T_1^*, T_2^*, T_3^*, \dots, T_0^*$ , equivalent to  $T_{n+1}, T_n, T_{n-1}, T_{n-2}, \dots, T_{n+1-i}$ , for  $i \in [0, 1, 2, 3, \dots, n+1]$ , therefore  $T_i^* = T_{n+1-i}$ . Here the constant time interval  $\delta$  is given as  $\delta = T_{i+1} - T_i$ .

Now that we have all the model assumptions and the integrability conditions, the task is to build the forward measure.

We first consider the longest maturity for the LIBOR rates that is  $L(., T^*)$  by backward induction. It will be shown later that this assumption is enough to make the  $L(., T^*)$  a martingale. Once this has been achieved we continue with the backwards induction to find all the other maturities for LIBOR up to the initial rates  $L(., T_0^*)$ .

The dynamics driving the terminal Lévy LIBOR at time  $T^*$  is  $L_t^{T^*}$ , which is given as the time inhomogeneous Lévy process. We also have the initial bond prices as initial values. With these initial conditions, we can also construct the initial term structure of the forward LIBOR values  $L(0, T_i)$  as

$$L(0, T_i) = \frac{1}{\delta} \left( \frac{B(0, T_i)}{B(0, T_i + \delta)} - 1 \right) > 0$$

**7.1.4 No Lévy LIBOR Rate at Time  $T^*$ .** This construction is done by backwards induction from  $[T^*, T_0^*]$ , the very first rate we will for the interval  $[T^*, T_1^*]$  and will be the rate at time  $T_1^*$ , and for  $[T_1^*, T_2^*]$  for time  $T_2$ . Hence at time rate time  $T^*$ , is not possible to get the rate, since we need the interval  $[T^* + \delta, T^*]$ , which we are not considering for the tenor structure.

**7.1.5 Forward Measure for Lévy LIBOR.** The change of measure we apply here is the forward measure. This is obtained from the risk neutral measure by changing the numeraire to the zero coupon bond prices process.  $P_T^*$  is the forward measure at time  $T^*$  and  $T_1^* < T^*$ . Let

$$\frac{dP_{T_1^*}^*}{dP_T^*} \bigg| \mathcal{F}_t = \frac{B(t, T_1^*)/B(0, T_1^*)}{B(t, T^*)/B(0, T^*)} = \frac{B(t, T_1^*)}{B(t, T^*)} \frac{B(0, T_1^*)}{B(0, T^*)} \quad (7.1.6)$$

Equation (7.1.6) can also be written as follows, using the conventional market formula as

$$1 + \delta L(t, T) = F(t, T, T + \delta) = \frac{B(t, T)}{B(t, T + \delta)}$$

$$\frac{dP_{T_1^*}^*}{dP_{T^*}^*} \bigg| \mathcal{F}_t = \frac{1 + \delta L(t, T_1^*)}{1 + \delta L(0, T_1^*)} = \frac{F(t, T_1^*, T^*)}{F(0, T_1^*, T^*)} \quad (7.1.7)$$

From equation (7.1.7), it is clear that,  $1 + \delta L$  represents the forward bond price which is known to be  $\frac{B(t, T_1^*)}{B(t, T^*)}$ . So we can evaluate  $L$  from  $dL$  by finding a Stochastic Differential Equation for the random variable  $1 + \delta L(t, T_1^*)$ . The model is discussed in section 7.2 below.

## 7.2 First Interval Construction of Lévy-LIBOR from $[T_1^*, T^*]$

We start the construction by defining the forward LIBOR rate with the longest maturity  $L(t, T_1^*)$ . Under the measure  $P_{T^*}$  as:

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) dL_s^{T^*}\right), \quad (7.2.1)$$

where the canonical decomposition of  $L_t^{T^*}$  is given by equation (7.1.1).

We assume that the drift component  $b_t$  of the given Lévy process  $L_t^{T^*}$  is

$$\int_0^t \lambda(s, T_1^*) b_s ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_R \left(e^{\lambda(s, T_1^*) z} - 1 - \lambda(s, T_1^*) z\right) \nu^{T^*, L}(ds, dz) \quad (7.2.2)$$

This specification of the drift component is motivated by Proposition (7.2.1) below.

**7.2.1 Proposition.** *If  $L(t, T_1^*)$  given by equation (7.2.1) is a martingale, it is sufficient that the drift is given by (7.2.2) above.*

*Proof.* We insert equation (7.1.1) into (7.2.1) to get the following:

$$L(t, T_1^*) = L(0, T_1^*) \exp\left(\int_0^t \lambda(s, T_1^*) (b_s ds + c_s^{1/2} dW_s^T + \int_R z \tilde{N}_1(ds, dz))\right)$$

To solve for  $dL(t, T_1^*)$ , we use the the Itô formula for Lévy process in theorem 6.2.9. Here the function  $f(t, X(t))$  is given as:

$$f(t, X(t)) = L(t, T_1^*) = L(0, T_1^*) \exp(X(t)) \quad (7.2.3)$$

$$\frac{\partial L(t, T_1^*)}{\partial(X(t))} = L(0, T_1^*) \exp(X(t)), \quad \frac{\partial^2 L(t, T_1^*)}{\partial(X(t))^2} = L(0, T_1^*) \exp(X(t))$$

where  $X_t$  is be given as

$$dX_t = \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{1/2} dW_t^T + \int_{\mathbb{R}} \lambda(t, T_1^*) z \tilde{N}_1(dt, dz)$$

The continuous part of  $X_t$  is  $X_t^c$  as

$$dX_t^c = \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{1/2} dW_t^T \quad (7.2.4)$$

$$(dX_t^c)^2 = \lambda^2(t, T_1^*) c_t dt \quad (7.2.5)$$

Now we find  $dL(t, T_1^*)$  using equation (6.2.13) from theorem 6.2.9 to get the following:

$$\begin{aligned} dL(t, T_1^*) &= L(0, T_1^*) \exp(X(t)) \left( \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{1/2} dW_t^T \right) \\ &\quad + L(0, T_1^*) \exp(X(t)) \left( \frac{1}{2} \lambda^2(t, T_1^*) c_t dt \right) \\ &\quad + (L(0, T_1^*) \exp(X(t))) \int_{\mathbb{R}} \left( e^{z \lambda(t, T_1^*)} - 1 - z \lambda(t, T_1^*) \right) \nu(dz) dt \\ &\quad + (L(0, T_1^*) \exp(X(t))) \int_{\mathbb{R}} (e^{z \lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \end{aligned} \quad (7.2.6)$$

We factor  $L(0, T_1^*) \exp(X(t))$  from equation (7.2.6) to get the following:

$$\begin{aligned} dL(t, T_1^*) &= L(0, T_1^*) \exp(X(t)) \left[ \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^T + \frac{1}{2} \lambda^2(t, T_1^*) c_t dt \right. \\ &\quad \left. + \int_{\mathbb{R}} \left( e^{z\lambda(t, T_1^*)} - 1 - z\lambda(t, T_1^*) \right) \nu(dz) dt + \int_{\mathbb{R}} (e^{z\lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \right] \end{aligned} \quad (7.2.7)$$

we use (7.2.3) to get the following:

$$\begin{aligned} \frac{dL(t, T_1^*)}{L(t^-, T_1^*)} &= \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^T + \frac{1}{2} \lambda^2(t, T_1^*) c_t dt \\ &\quad + \int_{\mathbb{R}} \left( e^{z\lambda(t, T_1^*)} - 1 - z\lambda(t, T_1^*) \right) \nu(dz) dt + \int_{\mathbb{R}} (e^{z\lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \end{aligned} \quad (7.2.8)$$

$$\begin{aligned} \frac{dL(t, T_1^*)}{L(t^-, T_1^*)} &= \left[ \lambda(t, T_1^*) b_t + \frac{1}{2} c_t \lambda^2(t, T_1^*) + \int_{\mathbb{R}} \left( e^{z\lambda(t, T_1^*)} - 1 - z\lambda(t, T_1^*) \right) \nu(dz) \right] dt \\ &\quad + \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^T + \int_{\mathbb{R}} (e^{z\lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \end{aligned} \quad (7.2.9)$$

For equation (7.2.9) to be a martingale then the drift component must be zero, that is

$$\lambda(t, T_1^*) b_t dt + \frac{1}{2} c_t \lambda^2(t, T_1^*) dt + \int_{\mathbb{R}} \left( e^{z\lambda(t, T_1^*)} - 1 - z\lambda(t, T_1^*) \right) \nu^{T^*, L}(dz) dt = 0.$$

If the drift component is zero then,

$$\frac{dL(t, T_1^*)}{L(t^-, T_1^*)} = \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_{\mathbb{R}} (e^{z\lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \quad (7.2.10)$$

We have to show that the two component of equation (7.2.10) are martingales. Then finally we can say the the whole equation is a martingale. From equation (7.2.10), the the Brownian component  $dW_t^{T^*}$  is the continuous martingale by standard Stochastic Calculus. The second term of equation (7.2.10) is a discontinuous martingale, but is not trivial by standard Stochastic Calculus but can be seen in [Cont and Tankov (2004)] that it is also a martingale.  $\square$

From equation (7.2.10), we can write the following:

$$dL(t, T_1^*) = L(t^-, T_1^*) \left( \lambda(t, T_1^*) c_t^{1/2} dW_t^{T^*} + \int_{\mathbb{R}} (e^{z\lambda(t, T_1^*)} - 1) \tilde{N}_1(dt, dz) \right) \quad (7.2.11)$$

$$dL(t, T_1^*) = L(t^-, T_1^*) dH \quad (7.2.12)$$

where

$$dH(t, T_1^*) = \lambda(t, T_1^*) c_t^{1/2} dW_t^{T^*} + \int_{\mathbb{R}} \left( e^{\lambda(t, T_1^*) z} - 1 \right) \tilde{N}_1(dt, dz) \quad (7.2.13)$$

From equation (7.2.12) we get the following:

$$L(t, T_1^*) = L(0, T_1^*)\mathcal{E}(H(t, T_1^*)) \quad (7.2.14)$$

where  $\mathcal{E}$  is the Stochastic exponential as shown in Proposition 6.2.4.

We find  $d(1 + \delta L(t, T_1^*))$  from equation (7.2.11) as follows: We refer to equation (7.1.7) here

$$d(1 + \delta L(t, T_1^*)) = \delta L(t, T_1^*)\lambda(t, T_1^*)c^{1/2}dW_t^T + \int_{\mathbb{R}} \delta L(t, T_1^*)(e^{\lambda(t, T_1^*)z} - 1)\tilde{N}_1(dt, dz) \quad (7.2.15)$$

we multiply and divide the terms on the right hand side of equation (7.2.15) by  $(1 + \delta L(t^-, T_1^*))$ :

$$\begin{aligned} d(1 + \delta L(t, T_1^*)) &= (1 + \delta L(t^-, T_1^*)) \left( \frac{\delta L(t^-, T_1^*)}{1 + \delta L(t^-, T_1^*)} \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} \right. \\ &\quad \left. + \int_{\mathbb{R}} \frac{\delta L(t^-, T_1^*)}{1 + \delta L(t^-, T_1^*)} (e^{\lambda(t, T_1^*)z} - 1) \tilde{N}_1(dt, dz) \right), \end{aligned} \quad (7.2.16)$$

Let

$$\ell(t^-, T_1^*) = \frac{\delta L(t^-, T_1^*)}{1 + \delta L(t^-, T_1^*)},$$

equation (7.2.16) becomes

$$d(1 + \delta L(t, T_1^*)) = (1 + \delta L(t^-, T_1^*, T^*)) \left( \ell(t^-, T_1^*) \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_{\mathbb{R}} \ell(t^-, T_1^*) (e^{\lambda(t, T_1^*)z} - 1) \tilde{N}_1(dt, dz) \right). \quad (7.2.17)$$

We define  $\alpha$  and  $\beta$  as follows and substitute into equation (7.2.17)

$$\alpha(t, T_1^*, T^*) = \ell(t^-, T_1^*) \lambda(t, T_1^*) \quad (7.2.18)$$

$$(\beta(t, z, T_1^*, T^*) - 1) = \ell(t^-, T_1^*) (e^{\lambda(t, T_1^*)z} - 1) \quad (7.2.19)$$

to get the following form for  $L(t, T_1^*)$

$$d(1 + \delta L(t^-, T_1^*)) = (1 + \delta L(t^-, T_1^*)) \left( \alpha(t, T_1^*, T^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_{\mathbb{R}} (\beta(t, z, T_1^*, T^*) - 1) \tilde{N}_1(dt, dz) \right) \quad (7.2.20)$$

where  $d(1) = 0$ .

We express the right hand side of equation (7.2.20) in the form  $d(1 + \delta L(t^-, T_1^*))$  because of the reasons explained in section (7.1.5).

**7.2.2 Note.** We note that from equation (7.2.20), which is the equation after the change of measure, has a different drift component from  $\lambda(t, T^*)b_t$  to the new drift in equation (7.2.2), the continuous volatility  $c^{1/2}$  remains the same. We also write the new Lévy measure as well as a new Brownian process after the change of measure.

**7.2.3 New Brownian motion.** From Theorem (3.3.7), when there is a change of measure, as shown in our case from  $[P_{T^*}, P_{T_1^*}]$ , the new Brownian motion

$$W_t^{T^*} = W_t^{T^*} - \int_0^t \alpha(s, T_1^*, T^*) c_s^{\frac{1}{2}} ds$$

is a martingale

**7.2.4 New Compensator.** From equation (7.2.20) the corresponding  $P_{T_1^*}$ -compensator of  $\mu^L$  is given by

$$\nu^{T_1^*, L}(dt, dz) = \beta(t, x, T_1^*, T^*) \nu^{T^*, L}$$

is what makes the jump process a martingale.

**7.2.5 New Martingale Part.** We define the forward martingale on the time interval  $[T_1^*, T^*]$ . This is done by using the Randon-Nikodym density as the stochastic exponential  $\mathcal{E}$  below

$$\frac{dP_{T_1^*}}{dP_{T^*}} = \mathcal{E}_{T_1^*}(M^1)$$

Hence the forward martingale part of the process (7.2.20), referring from proposition 6.2.8 is given by the following:

$$M_t^1 = \int_0^t (\alpha(s, T_1^*, T^*) c_s^{\frac{1}{2}} dW_s^{T^*} + \int_0^t \int_R (\beta(s, x, T_1^*, T^*) - 1) \tilde{N}_1(ds, dz)$$

We have the dynamics for  $[T_1^*, T^*]$  and now proceed to formulate the dynamics on  $[T_2^*, T_1^*]$  by backwards induction.

### 7.3 Second Interval of Construction of Lévy-LIBOR on $[T_2^*, T_1^*]$

Considering the new time frame from  $[T_2^*, T_1^*]$  by backwards induction we change the measure from  $P_{T_1^*}$  to  $P_{T_2^*}$  as follows

$$\frac{dP_{T_2^*}}{dP_{T_1^*}} \big| \mathcal{F}_t = \frac{1 + \delta L(t, T_2^*)}{1 + \delta L(0, T_2^*)} = \frac{F(t, T_2^*, T_1^*)}{F(0, T_2^*, T_1^*)} \quad (7.3.1)$$

The forward LIBOR rate  $L(t, T_2^*)$  under  $\mathbb{P}_{T_1^*}$  is given by:

$$L(t, T_2^*) = L(0, T_2^*) \exp \left( \int_0^t \lambda(s, T_2^*) dL_s^{T_1^*} \right) \quad (7.3.2)$$

where the Lévy component is given as

$$L_t^{T_1^*} = \int_0^t b_s^{T_1^*} ds + \int_0^t c_s^{\frac{1}{2}} dW_s^{T_1^*} + \int_0^t \int_R z \tilde{N}_2(ds, dz) \quad (7.3.3)$$

$$dL_t^{T_1^*} = b_t^{T_1^*} dt + c_t^{\frac{1}{2}} dW_t^{T_1^*} + \int_R z \tilde{N}_2(dt, dz) \quad (7.3.4)$$

where

$$\tilde{N}_2 = \mu^L - \beta(t, x, T_1^*, T^*) \nu^{T^*, L} \quad (7.3.5)$$

Now we substitute equation (7.3.4) into (7.3.2) to get the following:

$$L(t, T_2^*) = L(0, T_2^*) \left( \lambda(t, T_2^*) b_t dt + \lambda(t, T_2^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} + \int_R \left( e^{\lambda(t, T_2^*) z} - 1 \right) \tilde{N}_2(dt, dz) \right) \quad (7.3.6)$$

Where

$$L(0, T_2) = L(t^-, T_2^*)$$

We can find  $dL(t, T_2^*)$ , the same manner as shown in equation (7.2.10), as follows follows

$$dL(t, T_2^*) = L(t^-, T_2^*) \left( \lambda(t, T_2^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} + \int_R \left( e^{\lambda(t, T_2^*)z} - 1 \right) \tilde{N}_2(dt, dz) \right) \quad (7.3.7)$$

where the corresponding forward drift component of  $L(t, T_2^*)$  under the measure forward measure  $\mathbb{P}_{T_2^*}$  is given by

$$\int_0^t \lambda(s, T_2^*) b_s^{T_1^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_2^*) ds - \int_0^t \int_R \left( e^{z\lambda(s, T_2^*)} - 1 - z\lambda(s, T_2^*) \right) \nu^{T_1^*, L}(ds, dz) \quad (7.3.8)$$

We can now write equation (7.3.7) as follows

$$L(t, T_2^*) = L(0, T_2^*) \mathcal{E}(H(t, T_2^*)) \quad (7.3.9)$$

where

$$H(t, T_2^*) = \int_0^t \lambda(s, T_2^*) c_s^{\frac{1}{2}} dW_s^{T_1^*} + \int_0^t \int_R \left( e^{\lambda(s, T_2^*)z} - 1 \right) \tilde{N}_2(ds, dz) \quad (7.3.10)$$

We find  $\delta(dL(t, T_2^*))$  from equation (7.3.7) and multiply and divide the terms on the right hand side by  $1 + \delta L(t^-, T_2^*)$  as follows:

$$\begin{aligned} d(1 + \delta L(t, T_2^*)) &= \left( \frac{\delta L(t^-, T_2^*)}{1 + \delta L(t^-, T_2^*)} \lambda(t, T_2^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} \right. \\ &\quad \left. + \int_R \frac{\delta L(t^-, T_2^*)}{1 + \delta L(t^-, T_2^*)} (e^{\lambda(t, T_2^*)z} - 1) \tilde{N}_2(dt, dz) \right) \end{aligned} \quad (7.3.11)$$

$$\begin{aligned} d(1 + \delta L(t, T_2^*)) &= (1 + \delta L(t^-, T_2^*)) \left( \frac{\delta L(t^-, T_2^*)}{1 + \delta L(t^-, T_2^*)} \lambda(t, T_2^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} \right. \\ &\quad \left. + \int_R \frac{\delta L(t^-, T_2^*)}{1 + \delta L(t^-, T_2^*)} (e^{\lambda(t, T_2^*)x} - 1) \tilde{N}_2(dt, dz) \right) \end{aligned} \quad (7.3.12)$$

Let

$$\ell(t^-, T_2^*) = \frac{\delta L(t^-, T_2^*)}{1 + \delta L(t^-, T_2^*)}$$

$$\begin{aligned} d(1 + \delta L(t, T_2^*)) &= (1 + \delta L(t^-, T_2^*)) \left( \ell(t^-, T_2^*) \lambda(t, T_2^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} \right. \\ &\quad \left. + \int_R \ell(t^-, T_2^*) (e^{\lambda(t, T_2^*)z} - 1) \tilde{N}_2(dt, dz) \right) \end{aligned} \quad (7.3.13)$$

We substitute  $\alpha$  and  $\beta$  as follows into equation (7.3.13)

$$\alpha(t, T_2^*, T_1^*) = \ell(t^-, T_2^*)\lambda(t, T_2^*) \quad (7.3.14)$$

$$(\beta(t, x, T_2^*, T_1^*) - 1) = \ell(t^-, T_2^*)(e^{\lambda(t, T_2^*)x} - 1) \quad (7.3.15)$$

to get the following

$$d(1 + \delta L(t, T_2^*)) = (1 + \delta L(t^-, T_2^*)) \left( \alpha(t, T_2^*, T_1^*) c_t^{\frac{1}{2}} dW_t^{T_1^*} + \int_R (\beta(t, z, T_2^*, T_1^*) - 1) \tilde{N}_2(dt, dz) \right) \quad (7.3.16)$$

Here we can conclude that, equation (7.3.16), satisfies the forward LIBOR for the interval  $[T_2^*, T_1^*]$ .

We now establish a new martingale part, new Brownian motions and a new corresponding jump measure as a result of the change of measure.

**7.3.1 New Brownian Part.** From Theorem 3.3.7, when there is a change of measure, as shown in our case from  $P_{T_1^*}$  to  $P_{T_2^*}$ , the new Brownian motion

$$W_t^{T_2^*} = W_t^{T_1^*} - \int_0^t \alpha(s, T_2^*, T_1^*) c_s^{\frac{1}{2}} ds \quad (7.3.17)$$

is a martingale.

**7.3.2 New Compensator.** The  $P_{T_2^*}$  compensator

$$\nu^{T_2^*, L}(dt, dz) = \beta(t, z, T_2^*, T_1^*) \nu^{T_1^*, L}(dt, dz)$$

is a martingale.

**7.3.3 New Martingale.** Hence the martingale part of equation 7.3.16 is given by the following:

$$M_t^2 = \int_0^t (\alpha(s, T_2^*, T_1^*) c_s^{\frac{1}{2}} dW_s^{T_1^*} + \int_R (\beta(s, z, T_2^*, T_1^*) - 1) \tilde{N}_2(ds, dz).$$

So far we have established the forward LIBOR for two time periods, first for  $[T_1^*, T^*]$  and then second for  $[T_2^*, T_1^*]$  by backwards induction. We note that at the end of each period, we get a new martingale part, new Brownian motion and a new compensator.

## 7.4 Generalisation of Process for Backwards Induction

Continuing the process, in the time scale of our **tenor structure** we get the following general LIBOR rate:

$$L(t, T_j^*) = L(0, T_j^*) \exp \left( \int_0^t \lambda(s, T_j^*) dL_s^{T_{j-1}^*} \right) \quad (7.4.1)$$

Generally we have the following for the  $j$ th interval

$$\frac{dL(t, T_j^*)}{L(t^-, T_j^*)} = \lambda(t, T_j^*) c_t^{\frac{1}{2}} dW_t^{T_{j+1}^*} + \int_{\mathbb{R}} (e^{z\lambda(t, T_j^*)} - 1) \tilde{N}_{j+1}(dt, dz) \quad (7.4.2)$$

**7.4.1 General Forward Martingale.** The general martingale part of equation 7.4.1 is given by the following:

$$M_t^{j-1} = \int_0^t (\alpha(s, T_{j-1}^*, T_j^*) c_s^{\frac{1}{2}} dW_s^{T_{j-1}^*} + \int_0^t \int_R (\beta(s, x, T_{j-1}^*, T_j^*) - 1) \tilde{N}_{j-1}(ds, dx),$$

for the following change of measure

$$\frac{dP_{T_{j-1}^*}}{dP_{T_j^*}} = \mathcal{E}_{T_{j-1}^*}(M^{j-1})$$

**7.4.2 General Forward Drift.** Also the general drift component is given by

$$\int_0^t \lambda(s, T_j^*, T_{j-1}^*) b_s^{T_1^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_{j-1}^*) ds - \int_0^t \int_R \left( e^{z\lambda(s, T_{j-1}^*)} - 1 - z\lambda(s, T_{j-1}^*) \right) \nu^{T_j^*, L}(ds, dz). \quad (7.4.3)$$

**7.4.3 General Forward Brownian Part.** From Theorem 3.3.7, we have the corresponding general Brownian component of equation (7.3.16) as follows:

$$W_t^{T_{j-1}^*} = W_t^{T^*} - \int_0^t \alpha(s, T_{j-1}^*, T_j^*) c_s^{\frac{1}{2}} ds = \dots \quad (7.4.4)$$

$$(7.4.5)$$

Now, if we need to get the very last Brownian motion in the process after the backwards induction, we need to subtract all the extra terms that enters the drift term from the initial Brownian motion as follows:

$$W_t^{T_{i+1}^*} = W_t^{T^*} - \int_0^t \left( \sum_{j=i+1}^n \alpha(s, T_j, T_{j+1}) \right) c_s^{\frac{1}{2}} ds \quad (7.4.6)$$

**7.4.4 General Forward Compensator.** In terms of the compensator part, we get a product in terms of terms in the  $\beta$  for all the time intervals and the very last compensator for the terminal value  $\nu^{T^*, L}(dt, dz)$ . This is the case because any time we change an interval, we multiply the compensator by a corresponding  $\beta$  term.

We get  $\nu_{j-1}^{T^*}$  as the  $\mathbb{P}_{T_{j-1}^*}$  compensator of  $\mu^L$

$$\nu^{T_{j-1}^*, L}(dt, dz) = \beta(t, z, T_{j-1}^*, T_j^*) \nu^{T_j^*, L}(dt, dz) = \dots \quad (7.4.7)$$

$$\nu^{T_{i+1}^*, L}(dt, dz) = \prod_{j=i+1}^n \beta(t, z, T_j, T_{j+1}) \nu^{T^*, L}(dt, dz) \quad (7.4.8)$$

**7.4.5 General Forward Drift.** Also the general drift component is given by

$$\int_0^t \lambda(s, T_j^*, T_{j-1}^*) b_s^{T_1^*} ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_{j-1}^*) ds - \int_0^t \int_R \left( e^{z\lambda(s, T_{j-1}^*)} - 1 - z\lambda(s, T_{j-1}^*) \right) \nu^{T_j^*, L}(ds, dz). \quad (7.4.9)$$



**7.4.6 Note.** In this chapter, we have established how we will find the forward LIBOR given a time frame. We have done it for two time frames and generalised for our tenor structure. The question is, are there any problems associated with this method of calculating forward LIBORS? The answer is yes.

Earlier, we have discussed, that when we change the time frame for the calculations of LIBOR, we get a new martingale part, Brownian motion and a compensator.

We note that  $\nu^{T^*,L}$  is the compensator part from  $T_1^*$  to  $T^*$ . At the second interval  $T_2^*$  to  $T_1^*$   $\beta(t, x, T_1^*, T^*)\nu^{T^*,L}$  becomes our compensator and this continues at the end of the tenor as shown in (7.4.4).

In the next section, we do the construction of the Lévy process by using the terminal measure. The logic is to prevent having different Brownian component and different compensators as shown above. Also the specification of then drift component is addressed to get a martingale process for the Lévy LIBOR model, under the terminal measure.

## 7.5 Dynamics Using the same Terminal Measure

We use the same analogy used before in the case of LIBOR market model as shown in 5. We want to prevent having different Brownian parts and different jump measures. We express the process in one jump measure called the terminal measure and one Brownian component called the terminal Brownian motion. Then after using the terminal measure ideology, we have to ensure that the process is a martingale. For the process to be a martingale then, the drift component should be zero. We will then specify the drift component in such a way that we get a martingale.

We choose  $b_t^i$  so that  $L(t, T_i)$  defined previously in equation (7.4.1) has the form

$$L(t, T_i) = L(0, T_i) \exp \left( \int_0^t \lambda(s, T_i) b_s^i ds + \int_0^t \lambda(s, T_i) c^{1/2} dW_s^{T^*} + \int_0^t \lambda(s, T_i) z (\mu^L - \nu^{T^*})(ds, dz) \right) \quad (7.5.1)$$

Using the Itô's formula for jump process (theorem (6.2.9)) in equation (7.5.1), we get the following equation  $L(t, T_i)$ :

$$\begin{aligned} \frac{dL(t, T_i)}{L(t^-, T_i)} &= \left[ \lambda(t, T_i) b_t^i + \frac{1}{2} (\lambda(t, T_i))^2 c_t + \int_{\mathbb{R}} (e^{\lambda(t, T_i) z} - 1 - \lambda(t, T_i) z) \nu(dz) \right] dt \\ &\quad + \lambda(t, T_i) c^{1/2} dW_t^{T^*} + \int_0^R (e^{\lambda(s, T_i)} - 1) z (\mu^L - \nu^{T^*})(dt, dz). \end{aligned} \quad (7.5.2)$$

By equation (7.4.2) we have

$$\frac{dL(t, T_i)}{L(t^-, T_i)} = \lambda(t, T_i) c^{1/2} dW_t^{T_{i+1}} + \int_{\mathbb{R}} (e^{\lambda(t, T_i) z} - 1) (\mu^L - \nu^{T_{i+1}})(dt, dz) \quad (7.5.3)$$

where  $\tilde{N}_{i+1} = (\mu^L - \nu^{T_{i+1}})$ .

We can equate equation (7.5.2) and (7.5.3), since they are the same to get the following:

$$\begin{aligned}
& \lambda(t, T_i) c^{1/2} dW_t^{T_{i+1}} + \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1)(\mu^L - \nu^{T_{i+1}})(dt, dz) \\
&= \left[ \lambda(s, T_i) b_t^i + \frac{1}{2} (\lambda(t, T_i))^2 c_t + \int_R (e^{\lambda(t, T_i)z} - 1 - \lambda(t, T_i)z) \nu(dz) \right] dt \\
&+ \lambda(t, T_i) c^{1/2} dW_t^{T^*} + \int_0^R (e^{\lambda(s, T_i)z} - 1) z (\mu^L - \nu^{T^*})(dt, dz)
\end{aligned} \tag{7.5.4}$$

We simplify as:

$$\begin{aligned}
& \lambda(t, T_i) c^{1/2} dW_t^{T_{i+1}} + \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1)(\mu^L)(dt, dz) - \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1)(\nu^{T_{i+1}})(dt, dz) \\
&= \left[ \lambda(s, T_i) b_t^i + \frac{1}{2} (\lambda(t, T_i))^2 c_t + \int_R (e^{\lambda(t, T_i)z} - 1) z \nu(dz) - \lambda(t, T_i) z \nu(dz) \right] dt \\
&+ \lambda(t, T_i) c^{1/2} dW_t^{T^*} + \int_R (e^{\lambda(s, T_i)z} - 1) z (\mu^L)(dt, dz) - \int_R (e^{\lambda(s, T_i)z} - 1) z (\nu)(dt, dz)
\end{aligned} \tag{7.5.5}$$

we cancel out terms to get the following:

$$\begin{aligned}
& \left[ \lambda(s, T_i) b_t^i + \frac{1}{2} \lambda(t, T_i)^2 c_t - \lambda(t, T_i) z \nu(dz) \right] dt + \lambda(t, T_i) c^{1/2} [dW_t^{T^*} - dW_t^{T_{i+1}}] \\
&+ \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1)(\nu^{T_{i+1}})(dt, dz) = 0.
\end{aligned} \tag{7.5.6}$$

Make  $\lambda(s, T_i) b_t^i$  the subject

$$\begin{aligned}
\lambda(s, T_i) b_t^i dt &= -\frac{1}{2} \lambda(t, T_i)^2 c_t dt + \lambda(t, T_i) z \nu(dz) dt - \lambda(t, T_i) c^{1/2} [dW_t - dW_t^{T_{i+1}}] \\
&- \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1)(\nu^{T_{i+1}})(dt, dz).
\end{aligned} \tag{7.5.7}$$

We use the generalisations of the Brownian component and the jump measure as shown in equations (7.4.6) and (7.4.7) to get the following:

$$\begin{aligned}
\lambda(s, T_i) b_t^i dt &= -\frac{1}{2} \lambda(t, T_i)^2 c_t dt + \lambda(t, T_i) z \nu(dz) dt - \lambda(t, T_i) c_t \sum_{j=i+1}^n \ell(t, T_j) \lambda(t, T_j) dt \\
&- \int_{\mathbb{R}} (e^{\lambda(t, T_i)z} - 1) \prod_{j=i+1}^n (\beta(t, z, T_j, T_{j+1})) \nu(dt, dz)
\end{aligned} \tag{7.5.8}$$

we get

$$\begin{aligned} \lambda(t, T_i) b_t^i = & -\frac{1}{2} \lambda(t, T_i)^2 c_t - \lambda(t, T_i) c_t \sum_{j=i+1}^n \ell(t, T_j) \lambda(t, T_j) \\ & - \int_{\mathbb{R}} \left( e^{\lambda(t, T_i) z} - 1 \right) \prod_{j=i+1}^n \left( \beta(t, z, T_j, T_{j+1}) - \lambda(t, T_i) z \right) \nu(dz) \end{aligned} \quad (7.5.9)$$

we substitute the components of  $\beta$  and  $\ell$  into equation (7.5.9) to get the following:

$$\begin{aligned} \lambda(t, T_i) b_t^i = & -\frac{1}{2} \lambda(t, T_i)^2 c_t - \lambda(t, T_i) c_t \sum_{j=i+1}^n \frac{\delta L(t^-, T_j)}{1 + \delta L(t^-, T_j)} \lambda(t, T_j) \\ & - \int_{\mathbb{R}} \left( e^{\lambda(t, T_i) z} - 1 \right) \prod_{j=i+1}^n \left( 1 + \frac{\delta L(t^-, T_j)}{1 + \delta L(t^-, T_j)} (e^{\lambda(t, T_i) z} - 1) \right) - \lambda(t, T_i) z \right) \nu(dz) \end{aligned} \quad (7.5.10)$$

Equation (7.5.10) is the drift component when we use the terminal measure. This component of the drift makes the Lévy LIBOR model under the terminal measure a martingale.

It is apparent that using the terminal measure we have expressed the process for Lévy LIBOR in the terminal Brownian motion and in the terminal jump measure. We do not have many Brownian parts and compensators as compared to the construction by forward measures as shown in section 7.2 and 7.3. The apparent problem that is associated with using the terminal measure is the tractability of the drift component. The random term  $\ell(t, T_j^*)$  enters the drift and grows as a function of the tenor and since it is random, it is difficult to control. Also the random terms grows in the term  $\prod_{j=i+1}^n (\beta(t, z, T_j, T_{j+1}))$  as shown in equation (7.5.9) as exponential function of the tenor structure and makes the computations time consuming and complicated to truncate.

The random terms  $\ell(t, T_j^*)$  makes the Lévy process independent because, all the respective rates are dependent on each other. This destroys the first property of time inhomogeneous Lévy of independent increments as shown in definition 6.3.2. Hence what we are left with is a general semimartingale and not a Lévy process.

The standard remedy to the random drift term in the drift component is freezing drift approximation, where the random terms are replaced by their deterministic initial values. This makes the drift terms deterministic and hence the processes become time inhomogeneous Lévy processes again. As a result, the rates can be simulated in parallel since the dependencies are removed. The only pitfall with this approximation is that, “it is crude and does not yield acceptable results” as Papapantoleon and Skovmand (2010) put it.

**7.5.1 Chapter Conclusion.** We have established the Dynamics of Lévy LIBOR model, for their corresponding forward measures and the terminal measure. Dynamics under their forwards measures is complicated with different Brownian component and jump measures. Terminal measure is easy but coupled with complicated drift component. In the next chapter, we will look at the details of this problem and adopt some numerical methods to approximate the drift component. By so doing we can then manage the growth of the drift and also make our process tractable.

## 8. Picard Approximations and Cumulant Expansion for Lévy LIBOR

In this chapter, the main objective is to use Picard approximation and cumulant expansion to approximate the drift component in equation (7.5.10). The Picard approximation is done in section 8.1 and the cumulant expansion is done in section 8.2. We adopt the Picard approximation in order to make the drift deterministic and thus easier to evaluate. The resulting processes are time inhomogeneous Lévy processes after the approximation. The Picard approximation is done in the framework of Papantoleon and Skovmand (2010). The Cumulant process is used to approximate the integral term in the drift part in equation 7.5.9 by a summation. We use the notation and theory of the cumulant expansion as in Papantoleon et al. (2012).

We first state the The Picard-Lindelöf Theorem as follows:

**8.0.2 Theorem.** *The Picard-Lindelöf Theorem [Hunt and Kennedy (2004)]: Let  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  be a filtered probability space supporting a 1-dimensional Brownian motion  $W$  and suppose the 1-dimensional process  $Z$  is defined by*

$$Z_t = \int_0^t b_u du + \int_0^t \sigma_u dW_u$$

where  $b$  and  $\sigma$  are  $\mathbb{F}_t$ -predictable processes. Then, for all  $t \geq 0$ ,

$$E[Z_t^{*2}] \leq 2(4+t)E \left[ \int_0^t (|b_u|^2 + |\sigma_u|^2) du \right].$$

Now let  $\xi$  be an  $\mathcal{F}_0$ -measurable random variable and define the operator  $\mathcal{G}$  (or  $\mathcal{G}^\xi$  when we wish to emphasize the role of  $\xi$ ) by

$$(\mathcal{G}X)_t = \xi + \int_0^t b(X_u) du + \int_0^t \sigma(X_u) dW_u,$$

where now  $\sigma$  and  $b$  are Borel measurable functions on  $R^n$  satisfying the Lipschitz conditions

$$|b(x) - b(y)| \leq K |x - y|, \quad (8.0.1)$$

$$|\sigma(x) - \sigma(y)| \leq K |x - y|, \quad (8.0.2)$$

where  $K$  is a constant and  $X$  is a continuous adapted process taking values in  $R$ .

### 8.1 Picard Iteration

Here we consider the logarithms of the Lévy-LIBOR rates to derive the approximations. We adopt the logarithm of the terms as the summation of the terms are easy to handle than the multiplicative exponentials. We the log-LIBOR rates by  $Z$  as follows:

$$Z(t, T_i) := \log L(t, T_i),$$

$$Z(t, T_i) = Z(0, T_i) + \int_0^t \lambda(s, T_i) b(s, T_i) ds + \int_0^t \lambda(s, T_i) dH(s) \quad (8.1.1)$$

where

$$Z(0, T_i) = \log L(0, T_i), \quad \forall i \in \{1, \dots, N\}$$

and  $dH$  is defined in equation (7.3.10).

**8.1.1 The Linear SDE for log-LIBOR rates.** The dynamics of log-LIBOR can be described as the solution to the following SDE

$$\begin{aligned} dZ(t, T_i) &= \lambda(s, T_i) b(t, T_i; Z(t)) dt + \lambda(t, T_i) dH_t \\ Z(0, T_i) &= \log L(0, T_i) \end{aligned} \quad (8.1.2)$$

We introduce the term  $Z(t)$  in the drift component as  $b(t, T_i; Z(t))$  from equation (8.1.2) to show the dependencies of log-LIBOR rates.

**8.1.2 First Order Picard Iteration.** the first order Picard iteration is simply given by the initial values of the LIBOR rates

$$Z^{(1)}(t, T_i) = Z(0, T_i)$$

**8.1.3 Second Order Picard Approximation.** The the second order Picard approximation is defined when we have used the initial values in the process: The second order approximations for different  $T$  values considered are not dependent on each other since they are dependent on the first order approximation which is defined in the deterministic values.

$$Z^{(1)}(t, T_i) = Z(0, T_i) + \int_0^t \lambda(s, T_i) b(s, T_i; Z^{(0)}(s)) ds + \int_0^t \lambda(s, T_i) dH_s \quad (8.1.3)$$

$$= Z(0, T_i) + \int_0^t \lambda(s, T_i) b(s, T_i; Z(0)) ds + \int_0^t \lambda(s, T_i) dH_s \quad (8.1.4)$$

Here the drift term  $b(s, T_i; Z^{(0)}(s))$  is deterministic. We can clearly see that the random terms have been replaced with their initial values, this is the idea of Picard approximation. By doing this we have also used freezing the drift approximation in the drift component. We note that  $Z^{(1)}(t, T_i)$  is also a Lévy process with deterministic drift component.

We have approximated the semimartingale  $Z(t, T_i)$  using the second order Picard approximation. As a result we get a time-inhomogeneous Lévy process  $Z^{(1)}(t, T_i)$  in equation (8.1.3) as compared to a more general semimartingale in equation (8.1.2).

#### 8.1.4 Approximate Lévy Log-LIBOR Rate Using the Second Picard Approximation.

Since the drift component of the Lévy LIBOR process is the focus of approximation, we apply the second order Picard approximation in the drift component. Here  $\hat{Z}(t, T_i)$  represent the log-LIBOR rate using the second Picard iteration in the drift component. The process is given as

$$\hat{Z}(t, T_i) = Z(0, T_i) + \int_0^t \lambda(s, T_i) b(s, T_i; Z^{(1)}(s)) ds + \int_0^t \lambda(s, T_i) dH_s \quad (8.1.5)$$

the drift component  $b(s, T_i; Z^{(1)}(s))$  will be specified below in such a manner that will make the approximate Lévy Log-LIBOR process, in equation (8.1.5) a martingale. Also  $dH$  is defined in equation (7.3.10).

We assume that the drift component  $b_t$  of the given approximate Lévy log-LIBOR process  $\hat{Z}(t, T_i)$  is

$$\begin{aligned} \lambda(t, T_i)b_t^i = & -\frac{1}{2}\lambda(t, T_i)^2 c_t - \lambda(t, T_i)c_t \sum_{j=i+1}^n \ell(t, T_j)\lambda(t, T_j) \\ & - \int_{\mathbb{R}} \left( e^{\lambda(t, T_i)z} - 1 \right) \prod_{j=i+1}^n (\beta(t, z, T_j, T_{j+1}) - \lambda(t, T_i)z) \nu(dz), \end{aligned} \quad (8.1.6)$$

where we substitute  $L(t, T_j) = \exp(Z(t^-, T_j))$  in the specification of the terminal drift in equation (7.5.10) to get equation (8.1.6).

We substitute the components of  $\beta$  and  $\ell$  into equation (8.1.6) to get the following:

$$\begin{aligned} \lambda(t, T_i)b(s, T_i; Z^{(1)}(s)) = & -\frac{1}{2}\lambda(t, T_i)^2 c_t - \lambda(t, T_i)c_t \sum_{j=i+1}^n \frac{\delta(\exp(Z^{(1)}(t^-, T_j^*)))}{1 + \delta(\exp(Z^{(1)}(t^-, T_j^*)))} \lambda(t, T_j) \\ & - \int_{\mathbb{R}} \left( e^{\lambda(t, T_i)z} - 1 \right) \prod_{j=i+1}^n \left( 1 + \frac{\delta(\exp(Z^{(1)}(t^-, T_j^*)))}{1 + \delta(\exp(Z^{(1)}(t^-, T_j^*)))} (e^{\lambda(t, T_i)z} - 1) \right) \\ & - \lambda(t, T_i)z \nu(dz). \end{aligned} \quad (8.1.7)$$

The drift (8.1.7) makes the approximate Lévy log-LIBOR model under the terminal measure a martingale.

The resulting SDE  $\hat{Z}(t, T_i)$  depends on the Lévy process  $Z^{(1)}(t, T_l), i+1 \leq l \leq N$ , which are independent of each other. Hence all approximated LIBOR rates can be simulated independently.

The main reason for adopting approximations method is to remove dependencies of all respective rates in the process. Using Picard approximation, the resulting approximate log-LIBOR process is time inhomogeneous and also independent from each other, as a result there are no dependencies in the process.

**8.1.5 Log-LIBOR Rates can be Simulated in Parallel.** From equation (8.1.7) we see that, since subsequent rates are only dependent on the Lévy process  $Z^{(1)}(t, T_l)$ , the approximate rates from equation (8.1.5) can be simulated in parallel since there are no dependencies. The matrix of dependencies is shown below:

Table 8.1: Matrix of Dependencies for Log-LIBOR

$\hat{L}(t, T_1)$	$\hat{L}(t, T_2)$	$\cdots$	$\hat{L}(t, T_k)$	$\cdots$	$\hat{L}(t, T_{n-1})$	$\cdots$	$\hat{L}(t, T_n)$
$Z^1(t, T_n)$	$Z^1(t, T_n)$	$\cdots$	$Z^1(t, T_n)$	$\cdots$	$Z^1(t, T_n)$	$\cdots$	$Z^1(t, T_n)$
$Z^1(t, T_{n-1})$	$Z^1(t, T_{n-1})$	$\cdots$	$Z^1(t, T_{n-1})$	$\cdots$	$Z^1(t, T_{n-1})$	$\cdots$	$Z^1(t, T_{n-1})$
$Z^1(t, T_{n-2})$	$Z^1(t, T_{n-2})$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\cdots$	$\cdots$	$\cdots$	$Z^1(t, T_{k+1})$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$Z^1(t, T_3)$	$Z^1(t, T_3)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$Z^1(t, T_2)$	$Z^1(t, T_2)$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

## 8.2 Approximation of Integral in Lévy LIBOR Model using Cumulants

Generally equation (7.5.10) becomes the drift component when the construction is done using the terminal measure. An attempt to solve the Lévy LIBOR model from equation (7.5.1), using Monte Carlo simulations would have a stochastic component of integration at each time step. This is because the random term  $\frac{\delta_j L_j}{1+\delta L_j^-}$  appears under the integral sign.

The product term under the second component of equation (8.1.7) has the form:

$$\prod_{j=1}^n (1 + w_j) = (1 + w_1)(1 + w_2)(1 + w_3) \cdots (1 + w_n) \quad (8.2.1)$$

the number of terms on the right hand side of equation (8.2.1) when expanded is  $2^n$ .

To overcome this, we write the drift component in another form that uses cumulants for the Lévy process. This is achieved when we multiply the random quotients with the cumulant of the driving process. The detailed calculation can be found in Appendix A. Where the first order approximation is given as follows:

$$\lambda(t, T_i) b_t^i = \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_j^-}{1 + \delta_j L_j^-} (\kappa(\lambda_i + \lambda_j) - \kappa(\lambda_i) - \kappa(\lambda_j)) \quad (8.2.2)$$

**8.2.1 The reduction of terms after cumulant expansion.** After the expansion using cumulants, we have achieved a reduction in the number terms to be evaluated from exponential growth to quadratic growth.

Also we derive the second order expansion of the drift component as follows: Details of this approx-

imation are given in Appendix (A).

$$\begin{aligned}
\lambda(t, T_i) b_{ii}'' &= \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j-}}{1 + \delta_j L_{j-}} (\kappa(\lambda_i + \lambda_j) - \kappa(\lambda_i) - \kappa(\lambda_j)) \\
&+ \sum_{i+1 < k < l \leq N} \frac{\delta_k L_{k-}}{1 + \delta_k L_{k-}} \frac{\delta_l L_{l-}}{1 + \delta_l L_{l-}} (\hat{\kappa}(\lambda_i) + \lambda_k + \lambda_l) - \hat{\kappa}(\lambda_i + \lambda_l) - \hat{\kappa}(\lambda_l + \lambda_k) \\
&- \hat{\kappa}(\lambda_i) + \hat{\kappa}(\lambda_k) + \hat{\kappa}(\lambda_l)
\end{aligned} \tag{8.2.3}$$

where  $\kappa$  is given as below:

$$\hat{\kappa}(\lambda_i) = \int_{\mathbb{R}} \left( e^{\lambda_i^T x} - 1 - \lambda_i^T x \right) F(., dx) \tag{8.2.4}$$

Clearly from the approximations above we avoided numerical integration because there are no integrals in the approximation of the drift terms. Numerical results show that truncation of the drift term at the second order yields acceptable results.

**8.2.2 Chapter conclusion.** What we have achieved in this chapter is the approximation of the drift term under the terminal measure using the Picard approximation method. Here we used the second order Picard approximation as remedy. The resulting Log-Lévy process has the nice property of enabling terms to be simulated independently. We also use cumulants to reduce the computational time. The computational gain is the reduction from exponential growth to quadratic growth. We are now in a good position to look at the numerical simulation of the approximate log-Lévy rate using Picard approximation in the next chapter.



## 9. Numerical Illustration of Lévy LIBOR Model

The purpose of this chapter is to consider the efficiency of the Picard approximation employed in chapter 8 for the approximation of the drift component for the Lévy LIBOR model. Here we show the path simulation of LIBOR using the full numerical solution for Lévy LIBOR in section 9.1. In section 9.2, we show path simulations when we have used the Picard approximation for Lévy LIBOR using numerical integration in the drift component. Numerical integration is very time consuming, as will be shown later, so to increase the efficiency of the process we adopt cumulant expansion. Cumulant expansion of the drift term is dealt with in appenndix A in the framework of [Papapantoleon and Skovmand (2011)] to replace the numerical integration in the drift term with a summation. The first order and second order cumulant expansions for the Lévy LIBOR can be referred to in equations (8.2.2) and (8.2.3) respectively. Section 9.3 is the simulation of Lévy LIBOR with Picard approximation and cumulant expansion of the drift terms. Finally section 9.4 is when we consider a Brownian motion in the approximation instead of NIG pure jump for the jump part. Simulating the Lévy LIBOR model, we need to specify the parameters of the model. We first specify the jump term, and volatility  $\lambda(., T_i)$  of the model.

Here the jump term of the Lévy process we consider is given by Normal Inverse Gaussian process (NIG) with the following density function referring from equation (6.3.9):

$$NIG(x; \alpha, 0, 0, \delta) = \frac{\alpha\delta}{\pi} \exp \left\{ \delta\sqrt{\alpha^2} \right\} \frac{K_1(\alpha\sqrt{\delta^2 + (x)^2})}{\sqrt{\delta^2 + (x)^2}} \quad (9.0.1)$$

Where  $\beta = \mu = 0$  and  $K_1$  is the modified Bessel function of the third kind. In our specification we adopt  $\alpha = \delta = 1.5$ . The resulting NIG Process with the given parameters has mean zero and variance one. This specifications is what leads to a special kind of NIG Lévy jump process called a pure jump process and has the advantages of being a general jump model to work with as a result of the relaxation of  $\beta$  and  $\mu$ . The pure jump NIG process has the following canonical decomposition:

$$H = \int_0^\cdot \int_{\mathbb{R}} x(\mu^H - \nu)(ds, dx) \quad (9.0.2)$$

$\mu^H$  is the random measure of jumps  $H$ ,  $\nu$  is the compensator of  $\mu^H$ . In our simulation for Lévy LIBOR rate we use  $H$  as our specified jump component, the Lévy measure associated with  $H$  referring from equation (6.3.10) is given as

$$\nu_{NIG}(\alpha, 0, 0, \delta)dx = \frac{\delta\alpha}{\pi|x|} K_1(\alpha|x|)dx. \quad (9.0.3)$$

We use the notation  $\nu(dt, dx) = F(dx)dt = \nu_{NIG}(\alpha, 0, 0, \delta)dx$  as the Lévy measure throughout this work. Also we specify the cumulant generation function of the specified NIG distribution used here for  $\beta = \mu = 0$ , referring from the general equation for cumulant generating function of NIG, equation (6.3.6) is given as:

$$K(\mu) = \delta\alpha - \delta\sqrt{\alpha^2 - \mu^2}, \quad \mu \in \mathcal{C} \quad (9.0.4)$$

Now we have specified the jump process  $H$  and the jump measure  $\nu(dt, dx)$  to use, we now specify the volatilities  $\lambda(., T_i)$  of the Lévy LIBOR model as shown in equation (7.2.1). The volatilities used here are the same as those used by Klunge (2005) as:

$$\begin{aligned}
\lambda(., T_1) &= 0.20, & \lambda(., T_4) &= 0.19, & \lambda(., T_7) &= 0.18 \\
\lambda(., T_2) &= 0.17, & \lambda(., T_5) &= 0.16, & \lambda(., T_8) &= 0.15 \\
\lambda(., T_3) &= 0.14, & \lambda(., T_6) &= 0.13, & \lambda(., T_9) &= 0.12
\end{aligned}$$

We note that the  $\lambda$  values considered here are constant on a given time  $[0, T_i]$  interval. In a more general case  $\lambda$  can be taken to vary on the given interval.

We check the necessary condition to be satisfied by the chosen  $\lambda$  values as stated in assumption ?? under LR1. Here  $M = \alpha = 1.5$ .

$$\sum_{i=1}^n |\lambda(s, T_i)| < 1.5 \quad (9.0.5)$$

and

$$\lambda(s, T_i) < \frac{1.5}{2} \quad (9.0.6)$$

We conclude that, the  $\lambda$  values satisfied under the assumption EM in equation (7.1.5) under theorem (7.1.2). Now that we have all the necessary setting of the parameters, what we do next is the simulation of the approximated process  $\hat{Z}(t, T_i)$  from equation (8.1.5) in chapter 8.

## 9.1 Full Numerical Solution of Lévy LIBOR Model

Here we show the full numerical path simulation of Lévy LIBOR model as given in equation (8.1.1). Here the parameters of the pure jump NIG (Normal Inverse Gaussian Process) are given by  $\alpha = 1.5 = \delta$ , where the  $\sum_{i=1}^n \lambda(., i) = 0.7 < 1.5$  and  $dT = 0.5$ . Table 9.1 shows that modelling LIBOR with a Lévy process, depicts jumps in the rates as compared to table 5.2, when we model LIBOR with Brownian motion. We also note that, despite the introduction of jumps in the model, the model is coupled with a complicated drift component that is non-linear in the LIBOR rates and grows as a function of the tenor structure as shown in equation (7.5.10).

Table 9.1: Simulation of Full Numerical solution for Lévy LIBOR

$T$	$T_0 = 0$	$T_1 = 0.5$	$T_2 = 1.0$	$T_3 = 1.5$	$T_4 = 2.0$
Pure Jump (NIG)		0.081295	0.034904	-0.016614	0.0210566
$\lambda(., i)$ (NIG)		0.16	0.17	0.18	0.19
$L_0(T_n)$	0.055				
$L_1(T_n)$	0.055	0.0590006			
$L_2(T_n)$	0.055	0.05902313	0.06053667		
$L_3(T_n)$	0.055	0.05904405	0.06057962	0.05914669	
$L_4(T_n)$	0.055	0.05906338	0.0606193	0.05920396	0.06009951

In figure (9.1), we show some path simulate for the Lévy LIBOR.

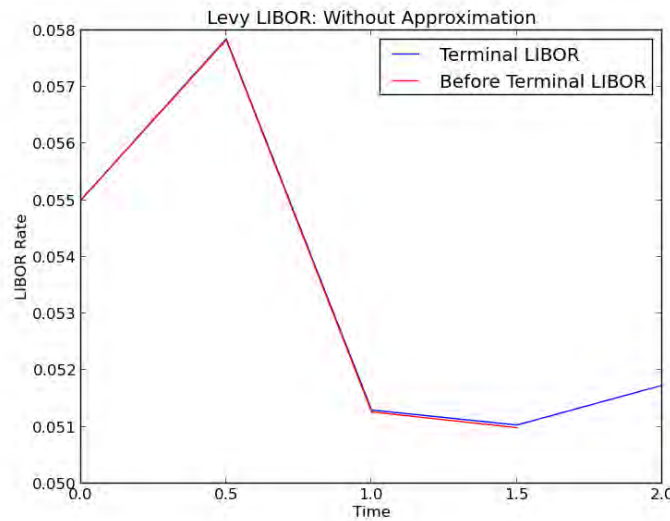


Figure 9.1: Path Simulation of Full Numerical Solution

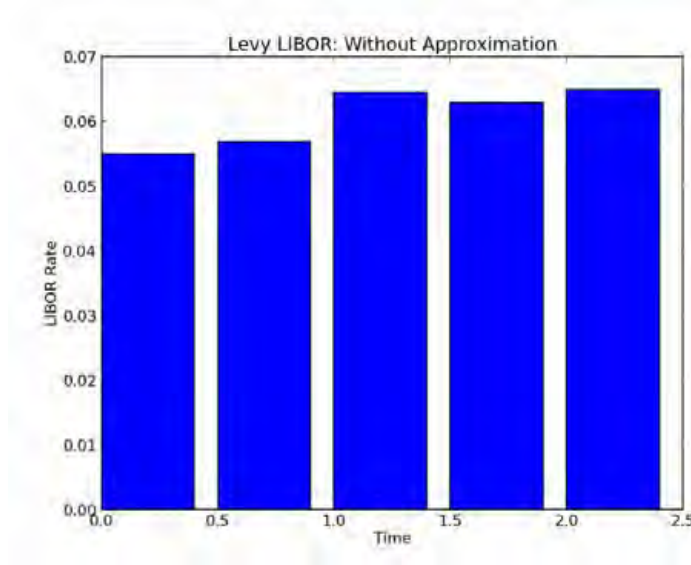


Figure 9.2: Bar Chat of Full Numerical Solution

## 9.2 Approximate Solution with Second Order Picard Approximation

Here we give simulations for Lévy LIBOR model using the second order Picard approximation as shown in equation (8.1.3). Using the second order Picard approximation for LIBOR is shown in equation (8.1.5), which results in a drift component that is not dependent on the previous rates and hence eliminates the dependencies of the rates as shown in equation (8.1.7) for the resulting drift component.

Table 9.2: Simulation of Approximate LIBOR Rate:  $n = 5, dT = 0.5, NIG(1.5, 0, 1.5, 0)$ 

LIBOR Rates	
$\hat{L}(T_0)$	0.055000
$\hat{L}(T_1)$	0.02382
$\hat{L}(T_2)$	0.054093
$\hat{L}(T_3)$	0.05041
$\hat{L}(T_4)$	0.080212

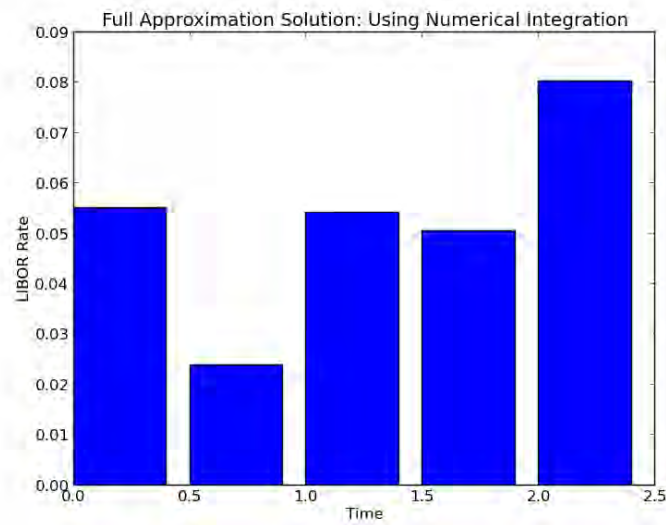


Figure 9.3: Approximate LIBOR Using Picard Approximation

### 9.3 Approximate Solution With Picard Approximation and Culmulant Expansion

The path simulation of LIBOR is considered when we use the second order cumulant expansion to truncate the numerical integration. The first and second cumulant expansion is shown in equations (8.2.2) and (8.2.3). We show some path simulations below:

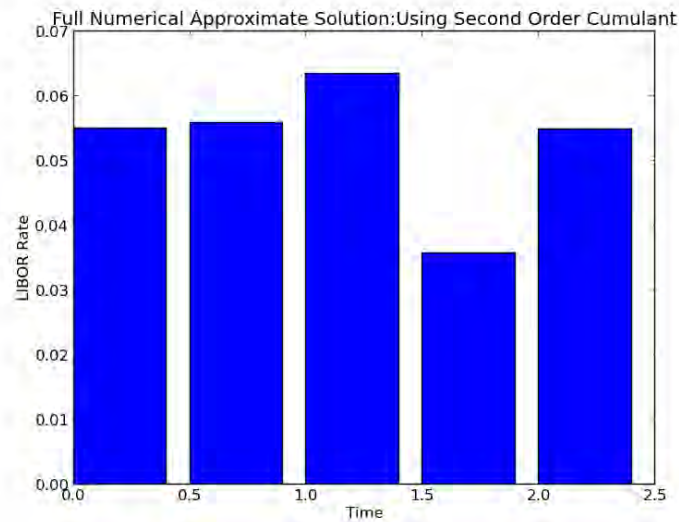


Figure 9.4: Full Approximation Solution with Cumulant Expansion: First Order

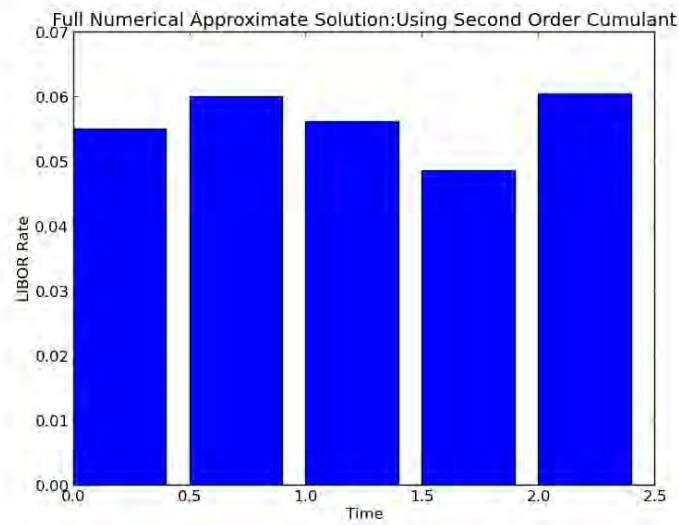


Figure 9.5: Full Approximation Solution with Second Order Cumulant Expansion

## 9.4 Solution for Lévy LIBOR with Brownian Motion as the Jump Part

Here we show the path simulation of Lévy LIBOR with Picard approximation and cumulant expansion but using a continuous Lévy process as the jump component. Here we use the Picard approximation and cumulant expansion with the exception of the jump part.

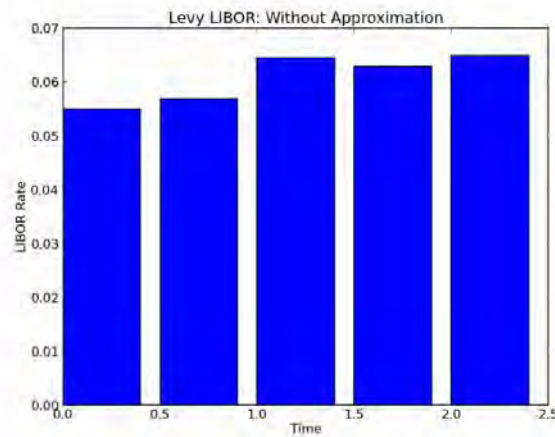


Figure 9.6: Simulation for Approximate LIBOR with Brownian Motion as Jump Part

## 9.5 Run Times of Methods for Approximating Lévy LIBOR Considered

Here we compare the run times of the various methods employed for simulating the Lévy LIBOR processes as shown in table 9.3. Approximation1 is when we consider the full numerical solution for the Lévy LIBOR model, Approximate2 is when we use Picard approximation, Approximation3 is when we employ both the second order Picard approximation and cumulant expansion. Finally approximation4 shows the run times when the model uses Brownian motion as the jump component instead of the NIG pure jump process employed. The running times in the table 9.4 show the average time for different time periods for tenor  $n = 4$  with different simulations periods considered.

Table 9.3: Run Time(Seconds) for The Four Approximation Methods Considered for Lévy LIBOR Model

Approximate1	Approximate2	Approximate3	Approximate4
0.077669	0.029692	0.001823	0.001841
0.076458	0.029958	0.001828	0.002095
0.076609	0.029201	0.001884	0.001842
0.076836	0.030006	0.001805	0.001855
0.076745	0.029213	0.001819	0.001878
0.077858	0.029149	0.001823	0.001838
0.077521	0.029063	0.001813	0.001839
0.076849	0.031359	0.001835	0.001849
0.078427	0.028881	0.002002	0.001847
0.079416	0.028919	0.001815	0.001842

Table 9.4: Average Run Time(Seconds)

Lévy Model	Average time(Seconds): n= 4
Approximate1	0.077439
Approximate2	0.029544
Approximate3	0.001845
Approximate4	0.001873

The run time for the different approaches considered shows that Second order Picard approximation of Lévy LIBOR with cumulant expansions gives the fastest results.

**9.5.1 Remark. Parallelism of LIBOR Rates:** Second order Picard approximation used here has the added advantage of being dependent only on the Lévy process  $Z^{(1)}(t, T_i)$ . Also  $Z^{(1)}(t, T_i)$  is dependent on the deterministic initial values of LIBOR rates given. As results the the LIBOR rates calculated using the second order Picard approximations can be simulated in parallel since they do not have dependencies.

**9.5.2 Chapter Conclusion.** In this chapter, we have established the numerical results for the Lévy LIBOR Market Model. The full numerical result of Lévy LIBOR model, the drift is still dependent on the LIBOR rates and we found this processes to be slow in terms of run time. Approximating the process using second order Picard approximate results in a model that is independent and

much more efficient run time. Combining Second order Picard approximation and the cumulant expansion, results in a processes that is independent from the LIBOR rates and also efficient. Substituting the jump component of the Lévy LIBOR model with Brownian motion results in a process that is efficient but without jumps in the market. From the four methods considered with agree that approximating Lévy LIBOR model with both second order Picard approximation and cumulant expansion is the ideal to give a model that is consistent with market prices and also computationally efficient. We also mention the added advantage of parallelism of rates as a result of the Picard approximation.



## 10. Conclusion

In this research, we have studied LIBOR Market Model (LMM) and The Lévy LIBOR Market Model (LLMM). Mostly in the literature LIBOR is modelled using LMM where Brownian motion is the driving underlying process. We studied that LMM is coupled with a nonlinear drift component despite the fact that it does not reflect the real market dynamics due to Brownian motion used. In chapter 5, We also given some path simulations for LMM, and also consider the pricing of cap under this model using the Markov Chain Monte Carlo Method (MCMC). MCMC method is not the ideal to use because it is time consuming. LLMM is the alternative model used to have a process that is more realistic with the introduction of jumps. Having a process that is more realistic in the framework of LLMM, we studied that is also coupled with a drift component that is non-tractable when we use the terminal measure.

In chapter 7 we provide a proof showing the resulting drift component when we use the terminal measure in the LLMM. The resulting drift term in equation (7.5.10), grows as a function of the tenor structure hence makes it difficult to track. In chapter 8, we have used the second order Picard approximation to the Lévy LIBOR model as seen in equation (8.1.5) where the truncated drift term is seen in equation (8.1.7). We also show the first order and second order cumulant expansion of the integral part of the drift term in equation (8.1.7) as shown in equations (8.2.2) and (8.2.3). In chapter 9 we have shown path simulations of LLMM for four different cases. Approximate1 when we simulate the full numerical solution of LIBOR, Approximate2 is the approximations as a result of second order Picard approximation, Approximate3 is the combination of Picard approximation and Cumulant expansion and finally Approximate4 is when we use Brownian motion in the jump part of the process for Lévy LIBOR instead of the pure jump NIG process.

In terms of run times for the four methods considered, it is found that Approximate3 has the highest efficiency as shown in table 9.3 and 9.4.

Finally we conclude that LIBOR Market Model is limited as compared to the general Lévy LIBOR model with more model flexibility as a result of jump component. Also the second order Picard Approximation of the Lévy LIBOR model with cumulant expansion gives a better approximation as compared to the other three methods considered.

Approximate Lévy LIBOR Model discussed here has a drift term that is independent from all other drift terms simulated as shown in equation (8.1.7), hence the rates are independent from each other. Further research will seek to apply some parallel computing schemes to simulate LIBOR. The Picard approximation alone yield great results as compared to the freezing the drift approximation mostly used. Applying parallelism is what adds great efficiency to the model since the model no longer has dependencies.

# Appendix A. Approximation of the Drift Component using Cumulant Expansion

In this appendix we show the approximation of the drift term as used in chapter 8: This is achieved using elementary symmetry polynomials.

In the first place, we consider the following expansion as follows:

$$\prod_{j=1}^l (1 + w_j) = (1 + w_1)(1 + w_2)(1 + w_3) \cdots (1 + w_l) \quad (\text{A.0.1})$$

When  $l = 2$ , we have the following:

$$\prod_{j=1}^2 (1 + w_j) = (1 + w_1)(1 + w_2) = 1 + w_1 + w_2 + w_1 w_2 \quad (\text{A.0.2})$$

We repeat for when  $l = 3$  to get the following:

$$\prod_{j=1}^3 (1 + w_j) = (1 + w_1)(1 + w_2)(1 + w_3) \quad (\text{A.0.3})$$

$$= 1 + w_1 + w_2 + w_3 + w_1 w_2 + w_1 w_3 + w_2 w_3 + w_1 w_2 w_3 \quad (\text{A.0.4})$$

$$= 1 + \sum_{1 \leq j \leq 3} w_j + \sum_{1 \leq j_1 < j_2 \leq 3} w_{j_1} w_{j_2} + \sum_{1 \leq j_1 < j_2 < j_3 \leq 3} w_{j_1} w_{j_2} w_{j_3} \quad (\text{A.0.5})$$

Considering equations (A.0.2) and (A.0.3) as examples we can define elementary symmetric polynomials  $S_p^l$  for  $l = 3$  as:

$$\sum_{p=1}^3 S_p^3(w_1, w_2, w_3) = \sum_{1 \leq j \leq 3} w_j + \sum_{1 \leq j_1 < j_2 \leq 3} w_{j_1} w_{j_2} + \sum_{1 \leq j_1 < j_2 < j_3 \leq 3} w_{j_1} w_{j_2} w_{j_3} \quad (\text{A.0.6})$$

$$(\text{A.0.7})$$

$S_p^l$  = Elementary symmetric polynomial of degree  $p$  in  $l$  variable here  $p$  ranges from  $(1 - 3)$ : With the use of this elementary polynomials equation (A.0.3) becomes:

$$\prod_{j=1}^3 (1 + w_j) = (1 + w_1)(1 + w_2)(1 + w_3) = 1 + \sum_{p=1}^3 S_p^l(w_1, w_2, w_3) \quad (\text{A.0.8})$$

Using the same analogy equation (A.0.1) becomes

$$\prod_{j=1}^l (1 + w_j) = 1 + \sum_{p=1}^l S_p^l(w_1, w_2, \dots, w_l) \quad (\text{A.0.9})$$

We will use this notion of elementary symmetric polynomial to the approximation of the drift term systematically:

We consider the original term of the drift component as follows:

$$B_i = \int_{\mathbb{R}^m} \left( (e^{\lambda_i^T x} - 1) \prod_{j=i+1}^N \left( 1 + \frac{\delta_j L_j (e^{\lambda_i^T x} - 1)}{1 + \delta_j L_j} \right) - \lambda_i^T x \right) F(., dx) \quad (\text{A.0.10})$$

$$B_i = \int_{\mathbb{R}^m} \left( (e^{\lambda_i^T x} - 1) \left( 1 + \sum_{p=1}^{N-1} S_p^{N-i} \left( \frac{\delta_{i+1} L_{i+1} (e^{\lambda_i^T x} - 1)}{1 + \delta_{i+1} L_{i+1}}, \dots, \frac{\delta_N L_N (e^{\lambda_i^T x} - 1)}{1 + \delta_N L_N} \right) \right) - \lambda_i^T x \right) F(., dx) \quad (\text{A.0.11})$$

Equation (A.0.11) can be written as follows:

$$B_i = \int_{\mathbb{R}^m} \left( e^{\lambda_i^T x} - 1 - \lambda_i^T x \right) F(., dx) + \sum_{p=1}^{N-1} \int_{\mathbb{R}} \left( e^{\lambda_i^T x} - 1 \right) \times S_p^{N-i} \left( \frac{\delta_{i+1} L_{i+1} (e^{\lambda_i^T x} - 1)}{1 + \delta_{i+1} L_{i+1}}, \dots, \frac{\delta_N L_N (e^{\lambda_i^T x} - 1)}{1 + \delta_N L_N} \right) F(., dx) \quad (\text{A.0.12})$$

We separate equation (A.0.12) as the following:

$$I = \int_{\mathbb{R}^m} \left( e^{\lambda_i^T x} - 1 - \lambda_i^T x \right) F(., dx) \quad (\text{A.0.13})$$

$$II = \sum_{p=1}^{N-1} \int_{\mathbb{R}} \left( e^{\lambda_i^T x} - 1 \right) \times S_p^{N-i} \left( \frac{\delta_{i+1} L_{i+1} (e^{\lambda_i^T x} - 1)}{1 + \delta_{i+1} L_{i+1}}, \dots, \frac{\delta_N L_N (e^{\lambda_i^T x} - 1)}{1 + \delta_N L_N} \right) F(., dx) \quad (\text{A.0.14})$$

We consider  $II$  for  $p \geq 1$

$$II = \sum_{i < j_1 < j_2 < \dots < j_p \leq N} \frac{\delta_{j_1} L_{j_1}}{1 + \delta_{j_1} L_{j_1}} \frac{\delta_{j_2} L_{j_2}}{1 + \delta_{j_2} L_{j_2}} \dots \frac{\delta_{j_p} L_{j_p}}{1 + \delta_{j_p} L_{j_p}} \times \int_{\mathbb{R}^m} (e^{\lambda_i^T x} - 1)(e^{\lambda_{j_1}^T x} - 1) \dots (e^{\lambda_{j_p}^T x} - 1) \quad (\text{A.0.15})$$

From equation (A.0.15), we can simplify the following:

$$III = (e^{\lambda_i^T x} - 1)(e^{\lambda_{j_1}^T x} - 1) \dots (e^{\lambda_{j_p}^T x} - 1) \quad (\text{A.0.16})$$

$$III = (-1)^{p+1} (1 - e^{\lambda_i^T x})(1 - e^{\lambda_{j_1}^T x}) \dots (1 - e^{\lambda_{j_p}^T x}) \quad (\text{A.0.17})$$

We use the elementary symmetric polynomial as shown in equation (A.0.9) and also take  $j_0 = i$  from equation (A.0.17) to get the following:

$$= (-1)^{p+1} \left[ 1 + \sum_{q=1}^{p+1} S_q^{p+1} \left( -e^{\lambda_{j_0}^T x}, \dots, -e^{\lambda_{j_p}^T x} \right) \right] \quad (\text{A.0.18})$$

and factorise out  $-1$  to get

$$= (-1)^{p+1} \left[ 1 + \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots \leq r_q \leq p} S_q^{p+1} \left( e^{\lambda_{j_{r_1}}^T x}, \dots, e^{\lambda_{j_{r_q}}^T x} \right) \right] \quad (\text{A.0.19})$$

Where

$$\sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} S_q^{p+1} \left( e^{\lambda_{jr_1}^T x}, \dots, e^{\lambda_{jr_q}^T x} \right) = \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} e^{\lambda_{jr_1}^T x}, \dots, e^{\lambda_{jr_q}^T x} \quad (\text{A.0.20})$$

then equation (A.0.18) becomes the following:

$$= (-1)^{p+1} [1 + *] \quad (\text{A.0.21})$$

where

$$* = \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} e^{\lambda_{jr_1}^T x}, \dots, e^{\lambda_{jr_q}^T x} \quad (\text{A.0.22})$$

Let

$$\sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} \left( 1 + \lambda_{jr_1}^T x + \lambda_{jr_2}^T x, \dots, \lambda_{jr_q}^T x \right) \quad (\text{A.0.23})$$

We add and subtract equation (A.0.23) to equation (A.0.22) to get the following:

$$\begin{aligned} * &= \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} \left( e^{\lambda_{jr_1}^T x}, \dots, e^{\lambda_{jr_q}^T x} - 1 - (\lambda_{jr_1}^T x + \lambda_{jr_2}^T x, \dots, \lambda_{jr_q}^T x) \right) \\ &+ \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} \left( 1 + \lambda_{jr_1}^T x + \lambda_{jr_2}^T x, \dots, \lambda_{jr_q}^T x \right). \end{aligned} \quad (\text{A.0.24})$$

Equation (A.0.16) is of order  $O(\|x\|^2)$  for any  $p \geq 1$ , hence the following must hold.

$$1 + \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} (1 + \lambda_{jr_1}^T x, \dots, \lambda_{jr_q}^T x) = 0 \quad (\text{A.0.25})$$

Hence equation A.0.16 becomes

$$III = (-1)^{p+1} \left[ \sum_{q=1}^{p+1} (-1)^q \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} e^{\lambda_{jr_1}^T x}, \dots, e^{\lambda_{jr_q}^T x} - 1 - (\lambda_{jr_1}^T x + \lambda_{jr_2}^T x, \dots, \lambda_{jr_q}^T x) \right] \quad (\text{A.0.26})$$

we substitute equation (A.0.26) into A.0.12 to get the following:

$$\begin{aligned} &= \int_{\mathbb{R}} \left( e^{\lambda_i^T x} - 1 - \lambda_i^T x \right) F(., dx) \\ &+ \sum_{p=1}^{N-i} \sum_{i \leq j_{r_1} \leq j_{r_2} \leq \dots, \leq j_{r_q} \leq N} \frac{\delta_{j_1} L_{j_1-}}{1 + \delta_{j_1} L_{j_1-}} \cdot \frac{\delta_{j_2} L_{j_2-}}{1 + \delta_{j_2} L_{j_2-}} \cdots \frac{\delta_{j_p} L_{j_p-}}{1 + \delta_{j_p} L_{j_p-}} \\ &\times \sum_{q=1}^{p+1} (-1)^{p+q+1} \sum_{0 \leq r_1 \leq r_2 \leq \dots, \leq r_q \leq p} \int_{\mathbb{R}} \left( e^{(\lambda_{jr_1} + \dots + \lambda_{jr_q})^T x} - 1 - (\lambda_{jr_1} + \dots + \lambda_{jr_q})^T x \right) F(., dx) \end{aligned} \quad (\text{A.0.27})$$

Let

$$\hat{\kappa}(\lambda_i) = \int_{\mathbb{R}} \left( e^{\lambda_i^T x} - 1 - \lambda_i^T x \right) F(., dx) \quad (\text{A.0.28})$$

$$\hat{\kappa}((\lambda_{jr_1} + \dots + \lambda_{jr_q})) = \int_{\mathbb{R}} \left( e^{(\lambda_{jr_1} + \dots + \lambda_{jr_q})^T x} - 1 - (\lambda_{jr_1} + \dots + \lambda_{jr_q})^T x \right) F(., dx) \quad (\text{A.0.29})$$

we substitute equation (A.0.28) and (A.0.29) into (A.0.27) to get the following:

$$\begin{aligned} &= \hat{\kappa}(\lambda_i) + \sum_{p=1}^{N-i} \sum_{i \leq j_1 \leq j_2 \leq \dots \leq j_p \leq N} \frac{\delta_{j_1} L_{j_1}^-}{1 + \delta_{j_1} L_{j_1}^-} \cdot \frac{\delta_{j_2} L_{j_2}^-}{1 + \delta_{j_2} L_{j_2}^-} \cdots \frac{\delta_{j_p} L_{j_p}^-}{1 + \delta_{j_p} L_{j_p}^-} \\ &\times \sum_{q=1}^{p+1} (-1)^{p+q+1} \sum_{0 \leq r_1 \leq r_2 \leq \dots \leq r_q \leq p} \hat{\kappa}((\lambda_{jr_1} + \dots + \lambda_{jr_q})) \end{aligned} \quad (\text{A.0.30})$$

### A.0.3 First Order Expansion, $p = 1$ .

$$B_i = \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j^-}}{1 + \delta_j L_{j^-}} \sum_{q=1}^2 (-1)^q \sum_{i \leq r_1 \leq r_2 \leq \dots \leq r_q \leq 1} \hat{\kappa}(\lambda_{jr_1} + \dots + \lambda_{jr_q}) + O(|L|)^2 \quad (\text{A.0.31})$$

Let

$$\sum_{q=1}^2 (-1)^q \sum_{i \leq r_1 \leq r_2 \leq \dots \leq r_q \leq 1} \hat{\kappa}(\lambda_{jr_1} + \dots + \lambda_{jr_q}) = - \sum_{0 \leq r_1 \leq 1} \hat{\kappa}(\lambda_{r_1}) + \sum_{0 \leq r_1 \leq r_2 \leq 1} \hat{\kappa}(\lambda_{r_1}) \quad (\text{A.0.32})$$

$$= -\hat{\kappa}(\lambda_{j_0}) - \hat{\kappa}(\lambda_{j_1}) + \hat{\kappa}(\lambda_{j_0} + \lambda_{j_1}) \quad (\text{A.0.33})$$

We substitute (A.0.33) into (A.0.31) to get the following:

The first order expansion is then given as:

$$B'_i = \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j^-}}{1 + \delta_j L_{j^-}} (\hat{\kappa}(\lambda_i + \lambda_j) - \hat{\kappa}(\lambda_i) - \hat{\kappa}(\lambda_j)) \quad (\text{A.0.34})$$

Hence the first approximation of the drift term in the first order is given as

$$b'_i = \kappa(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j^-}}{1 + \delta_j L_{j^-}} (\kappa(\lambda_i + \lambda_j) - \kappa(\lambda_i) - \kappa(\lambda_j)) \quad (\text{A.0.35})$$

**A.0.4 Second Order Expansion,  $p = 2$ .** We use the same approach to find the second order approximation of the drift term as

$$\begin{aligned}
B_i &= \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j-}}{1 + \delta_j L_{j-}} (\hat{\kappa}(\lambda_i + \lambda_j) - \hat{\kappa}(\lambda_i) - \hat{\kappa}(\lambda_j)) \\
&+ \sum_{i+1 < k < l \leq N} \frac{\delta_k L_{k-}}{1 + \delta_k L_{k-}} \frac{\delta_l L_{l-}}{1 + \delta_l L_{l-}} (\hat{\kappa}(\lambda_i + \lambda_k + \lambda_l) - \hat{\kappa}(\lambda_i + \lambda_k) - \hat{\kappa}(\lambda_i + \lambda_l) - \hat{\kappa}(\lambda_l + \lambda_k) \\
&- \hat{\kappa}(\lambda_i) + \hat{\kappa}(\lambda_k) + \hat{\kappa}(\lambda_l)) + O(|L|)^3
\end{aligned} \tag{A.0.36}$$

Equation (A.0.36) leads to the following second order approximations:

$$\begin{aligned}
b''_{ii} &= \hat{\kappa}(\lambda_i) + \sum_{i < j \leq N} \frac{\delta_j L_{j-}}{1 + \delta_j L_{j-}} (\kappa(\lambda_i + \lambda_j) - \kappa(\lambda_i) - \kappa(\lambda_j)) \\
&+ \sum_{i+1 < k < l \leq N} \frac{\delta_k L_{k-}}{1 + \delta_k L_{k-}} \frac{\delta_l L_{l-}}{1 + \delta_l L_{l-}} (\hat{\kappa}(\lambda_i) + \lambda_k + \lambda_l) - \hat{\kappa}(\lambda_i + \lambda_l) - \hat{\kappa}(\lambda_l + \lambda_k) \\
&- \hat{\kappa}(\lambda_i) + \hat{\kappa}(\lambda_k) + \hat{\kappa}(\lambda_l))
\end{aligned} \tag{A.0.37}$$

## Appendix B. Exponential Transform of a Lévy Process

Here we show that for a given Lévy process, we can find the stochastic exponential component using Lemma (2.6) from [Kallsen and Shiryaev \(2002a\)](#). This prove is a motivation to understand the term  $H$  in equation [7.2.13](#) and other places throughout this work.

The formula for the exponential transform is given as

$$\tilde{X}_t = X_t + \frac{1}{2}\langle X^c, X^c \rangle + \int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \mu^X(ds, dx) \quad (\text{B.0.1})$$

$$\Delta X_t = \lambda(s, T_1^*) dL_s^{T^*}$$

where  $L_s^{T^*}$  is given in equation [\(7.1.1\)](#), so we compute the following:

$$X_t = \int_0^t \lambda(s, T_1^*) dL_s^{T^*} \quad (\text{B.0.2})$$

$$dX_t^c = \lambda(t, T_1^*) b_t dt + \lambda(t, T_1^*) c_t^{\frac{1}{2}} dW_t^T \quad (\text{B.0.3})$$

$$(dX_t^c)^2 = \lambda^2(t, T_1^*) c_t dt \quad (\text{B.0.4})$$

Here we have to find the equivalence of  $\int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \mu^X(ds, dx)$

$$\int_0^t \int_{\mathbb{R}} (e^x - 1 - x) \mu^X(ds, dx) = \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z) \mu^L(ds, dz) \quad (\text{B.0.5})$$

We now substitute equation [\(B.0.2, B.0.3, B.0.5\)](#) into [\(B.0.1\)](#) to get the following:

$$\tilde{X}_t = \int_0^t \lambda(s, T_1^*) dL_s^{T^*} + \frac{1}{2} \int_0^t \lambda^2(s, T_1^*) c_s ds + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z) \mu^L(ds, dz) \quad (\text{B.0.6})$$

Where  $\int_0^t \lambda(s, T_1^*) dL_s^{T^*}$  and  $\int_0^t \lambda(s, T_1^*) b_s ds$  are given by

$$\int_0^t \lambda(s, T_1^*) dL_s^{T^*} = \int_0^t \lambda(s, T_1^*) b_s ds + \int_0^t \lambda(s, T_1^*) c_s^{\frac{1}{2}} dW_s^{T^*} + \int_R \int_0^t \lambda(s, T_1^*) z \tilde{N}_1(ds, dz) \quad (\text{B.0.7})$$

$$\int_0^t \lambda(s, T_1^*) b_s ds = -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_R (e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z) \nu^{T^*, L}(ds, dz) \quad (\text{B.0.8})$$

Hence

$$\begin{aligned} \int_0^t \lambda(s, T_1^*) dL_t^{T^*} &= -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_R \left( e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z \right) \nu^{T^*, L}(ds, dz) \\ &\quad + \int_0^t \lambda(s, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_R \int_0^t \lambda(s, T_1^*) z \tilde{N}_1(dt, dz) \end{aligned} \quad (\text{B.0.9})$$

We substitute equation (B.0.9) into (B.0.6) to get the following:

$$\begin{aligned} \tilde{X}_t &= -\frac{1}{2} \int_0^t c_s \lambda^2(s, T_1^*) ds - \int_0^t \int_R \left( e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z \right) \nu^{T^*, L}(ds, dz) \\ &\quad + \int_0^t \lambda(s, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_R \int_0^t \lambda(s, T_1^*) z \tilde{N}_1(dt, dz) \\ &\quad + \frac{1}{2} \int_0^t \lambda^2(s, T_1^*) c_s ds + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z) \mu^L(ds, dz) \end{aligned} \quad (\text{B.0.10})$$

we simplify to get the following:

$$\begin{aligned} \tilde{X}_t &= \int_0^t \lambda(s, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_R \int_0^t \lambda(s, T_1^*) z (\mu^L - \nu^{T_1^*})(dt, dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1 - \lambda(s, T_1^*)z) (\mu^L - \nu^{T_1^*})(ds, dz) \\ &\quad - \int_R \int_0^t \lambda(s, T_1^*) z (\mu^L - \nu^{T_1^*})(dt, dz) \end{aligned} \quad (\text{B.0.11})$$

where  $N_1(dt, dz) = \mu^L - \nu^{T_1^*}$

$$\begin{aligned} \tilde{X}_t &= \int_0^t \lambda(s, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1) (\mu^L - \nu^{T_1^*})(ds, dz) \\ &= \int_0^t \lambda(s, T_1^*) c_t^{\frac{1}{2}} dW_t^{T^*} + \int_0^t \int_{\mathbb{R}} (e^{\lambda(s, T_1^*)z} - 1) \tilde{N}_1(ds, dz) \end{aligned} \quad (\text{B.0.12})$$



# Appendix C. Algorithm and Python Code for Simulating Normal Inverse Gaussian Process

Here we give the algorithm for the simulation of Normal Inverse Gaussian Process as can be seen in [Kitchen \(2009\)](#) as well as a python computer program for simulating the process.

## C.1 Algorithm for NIG process

Here we first have the simulation of inverse Gaussian random variable, inverse Gaussian process and finally the NIG process.

**C.1.1 We first use the following algorithm to simulate the inverse Gaussian random numbers  $IG(a, b)$  using the following:**

1. Generate a standard normal random number  $v$ .
2. Set  $y = v^2$ .
3. Set  $x = \left(\frac{a}{b}\right) + \frac{y}{(2b^2)} + \frac{\sqrt{4aby + y^2}}{2b^2}$ .
4. Generate a uniform random number  $u$ .
5. If  $u \leq \frac{a}{(a+xb)}$ , then return the number  $x$  as the  $IG(a, b)$  random number, else return  $\frac{a^2}{(b^2x)}$  as the  $IG(a, b)$  random number.

**C.1.2 Inverse Gaussian Process Simulation  $IG(a\Delta t, b)$ .**

1. Generate  $n$  independent  $IG(a\Delta t, b)$  random numbers  $i_n, n \geq 1$ .
2. Set initial process value to  $X_0^{IG} = 0$ .
3. Iterate path by  $X_{n\Delta t}^{(IG)} = X_{(n-1)\Delta t}^{(IG)} + i_n$ .

**C.1.3 Simulating the Normal inverse Gaussian process  $IG(a, b)$ .**

1. Simulate each state of an inverse Gaussian process  $X(IG) = Xt, t \leq 0$  at time points  $n\Delta t, n = 0, 1, \dots$  using the algorithm above with  $a = 1$  and  $b = \delta\sqrt{\alpha^2 - \beta^2}$ .
2. Difference each consecutive state of  $dt_{n\Delta t} = X_{n\Delta t}^{IG} - X_{(n-1)\Delta t}^{IG}$ .
3. Simulate time change of a standard Brownian motion  $W = \{W_t, t \geq 0\}$  by
  - Simulate  $n$  independent standard normal random variables  $\nu, n > 0$ .
  - Set  $W_0 = W_{X_0^{(IG)}} = 0$ .
4.  $W_{n\Delta t} = W_{(n-1)\Delta t} + \sqrt{dt_{n\Delta t}}\nu_n$ .
4.  $X_{n\Delta t}^{NIG} = \beta\delta^2 X_{n\Delta t}^{IG} + \delta W_{n\Delta t}$ .

## C.2 Python Code for NIG Process

```

from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import random
#-----#
#This function computes the Inverse Guassian Random Random variable      #
#-----#
def IGR(a, b, n):
    IGR = []
    while len(IGR) < n:
        v = random.gauss(0,1)
        u = random.uniform(0,1)
        y = v**2
        x = (a/b) + y/(2*b**2) + np.sqrt(4*a*b*y + y**2)/(2*b**2)
        if u <= a/(a+x*b):
            IGR.append(x)
            # return NIG.append(x)
        else: IGR.append(a**2/(b**2*x))
    return IGR
#-----#
#This function computes the Inverse Guassian Process                      #
#-----#
def IGP(a, b, n):
    dT = 1/1000
    x = np.zeros(n+1)
    B = np.zeros((n+1,n+1))
    T = np.arange(0, 1 + dT, dT)
    v = IGR(a*dT,b,n+1)

    for i in range(1,n+1):
        x[i] = x[i-1] + v[i-1]
    return T, x
if __name__ == '__main__':
    #Constants
    n = 1000
    a = 1
    alpha = 50
    beta = -5
    delta = 1
    b = delta*np.sqrt(alpha**2 - beta**2)
    dT = 1/1000
    T = np.arange(0, (n+1) * dT, dT)
    #-----#
    #Here we computes the Normal Inverse Guassian Process                #
    #-----#

```

---

```

W = np.zeros(n+1)
for k in range(n):
    W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
#Simulate each state of an inverse Gaussian process
IGP1 = IGP(a,b,n)[1]
#Difference each consecutive state of inverse Gaussian process
diff = [IGP1[i] - IGP1[i-1] for i in range(1, len(IGP1))]
#Simulate time change of a standard Brownian motion
for i in range(1,n+1):
    W[i] = W[i-1] + np.sqrt(diff[i-1])*random.gauss(0,1)
#-----#
#X = NIG = N(alpha,beta,delta,mean)                                     #
#-----#
X = np.zeros(len(IGP1))
for i in range(len(IGP1)):
    X[i] = beta*delta**2*IGP1[i] + delta*W[i]
#Plotting NIG increaments
plt.plot(T[:-1],X)
plt.xlabel('Time')
plt.title('Normal Inverse Guassian Process N(50,-5,1)')
plt.ylabel('Normal Inverse Guassian Process increments')
plt.show()
plt.show()

```

## Appendix D. Python Code for LIBOR Market Model

We provide a python code for the full numerical solution of dynamical equation of the LIBOR Market Model as shown in equation (5.4.16) or (5.4.17). We also plot the path simulation for the terminal LIBOR and the rates just before the terminal LIBOR using the same Brownian motion.

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import random
n = 1000          #The tenor length
dT = 0.5          #Time difference
sigma = 0.15      #Constant volatility
delta = 0.5       #Constant tenor interval
#-----#
#Here we initialise the Browian Motion(W), Bond, Time interval(T), LIBOR (L) #
#-----#
W = np.zeros(n+1)
B = np.zeros((n+1,n+1))
T = np.arange(0, (n+1) * dT, dT)
L = np.zeros([n+1, n+1])
L[:,0] = 0.055 * np.ones(n+1)

#Simulating Brownian Motion
for k in range(n):
    W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
#-----#
#Simulating LIBOR Rates #
#-----#
for col in range(n+1):
    for row in range(col+1,n+1):
        dummy = 0
        for k in range(row+1,n+1):
            dummy += (sigma*delta*L[k,col])/(1+delta*L[k,col])
        L[row][col+1] = L[row][col]*np.exp((-dummy*sigma - 1/2*sigma**2)*dT
            + sigma*(W[col+1]-W[col]))
#-----#
#Plotting LIBOR Rates #
#-----#
plt.figure()
plt.plot(T[:-1], L[-2,:-1], color = 'blue', label='Before Terminal LIBOR')
plt.plot(T, L[-1,:], color = 'red', label='Terminal LIBOR')
plt.xlabel('Time')
plt.ylabel('LIBOR Rate')
plt.title('Superimposed Terminal LIBOR and Before Terminal LIBOR')
```

```
plt.legend()  
plt.show()
```

# Appendix E. Python Program For Monte Carlo Method for LIBOR Market Model: Calculating Caps and Floors

We provide the simulation of LIBOR Market Model as in equation (5.4.16) or (5.4.17). Here we price Caps and Floors as discussed in section 5.7.1.

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import random
N = 100000000    #The number of times for the MonteCarlo Loop
n = 4            #The tenor length
dT = 0.5         #Time difference
sigma = 0.15     #Constant volatility
delta = 0.5      #Constant tenor interval
VC = []          #The Cap values
VF = []          #The Cap values
#-----#
#Here we initialise the Browian Motion(W), Bond, time interval(T), LIBOR (L)  #
#-----#

for i in range(N):
    W = np.zeros(n+1)
    B = np.zeros((n+1,n+1))
    T = np.arange(0, (n+1) * dT, dT)
    L = np.zeros([n+1, n+1])
    L[:,0] = 0.05 * np.ones(n+1)
#Simulating Brownian Motion

    for k in range(n):
        W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
#-----#
#Simulating LIBOR Rates                                     #
#-----#

    for col in range(n+1):
        for row in range(col+1,n+1):
            dummy = 0
            for k in range(row+1,n+1):
                dummy += (sigma*delta*L[k,col])/(1+delta*L[k,col])
            L[row][col+1] = L[row][col]*np.exp((-dummy*sigma - 1/2*sigma**2)*dT
            + sigma*(W[col+1]-W[col]))

#-----#
```

---

```

#Calculating numeraire boased bond price                                     #
#-----#

    for col in range(n+1):
        for row in range(col,n+1):
            Prod = 1
            for k in range(col, row+1):
                Prod = Prod * (1 + delta * L[k][col])**-1
            B[row][col] = Prod
#-----#
#Calculating the Value of cap and floor                                     #
#-----#

K = 0.05                                #Strike Price
C = np.zeros((n+1,n+1))
F = np.zeros((n+1,n+1))
for col in range(n):
    for row in range(col,n+1):
        for k in range(col, row+1):
            if (L[row][col] -K) > 0:
                C[row][col+1] = (L[row][col] - K)/B[-1][col+1] #Cap Value
            if (K - L[row][col]) > 0:
                F[row][col+1] = (K - L[row][col] )/B[-1][col+1] #Floor Value

VC.append(sum(sum(C)))                #Different Cap values
VF.append(sum(sum(F)))                #Different Floor values
meanC = np.average(VC)                #Average Value of Cap
stdC = np.std(VC)                    #Standard Deivation of Cap
meanF = np.average(VF)                #Average Value of Floor
stdF = np.std(VF)                    #Standard Deivation of Floor

```

# Appendix F. Python Program For Levy LIBOR Model without Approximation

Here we give a python program showing the numerical implementation of Lévy LIBOR model discussed in chapter 7. The simulation is for equation (7.5.1) the full numerical solution of Lévy LIBOR model using the terminal measure as well as using numerical integration without approximations employed.

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import random
import scipy.integrate as int
from scipy.special import kv
import itertools

#-----#
#This function computes the Inverse Guassian Random Random variable #
#-----#
def IGR(a, b, n):
    IGR = []
    while len(IGR) < n:
        v = random.gauss(0,1)
        u = random.uniform(0,1)
        y = v**2
        x = (a/b) + y/(2*b**2) + np.sqrt(4*a*b*y + y**2)/(2*b**2)
        if u <= a / (a+x*b):
            IGR.append(x)
        else: IGR.append(a**2/(b**2*x))
    return IGR

#-----#
#This function computes the Inverse Guassian Process #
#-----#
def IGP(a, b, n):
    dT = 0.5
    x = np.zeros(n+1)
    B = np.zeros((n+1,n+1))
    T = np.arange(0, 1 + dT, dT)
    v = IGR(a*dT,b,n+1)
    for i in range(1,n+1):
        x[i] = x[i-1] + v[i]
    return T, x

#-----#
#This function computes the the integral of the NIG process and the lambda_values generated#
#-----#
def jumppart(lamda_values, diffH):
```



```

    Jump1 = []
    for x, y in itertools.izip(lamda_values, diffH):
        Jump1.append(x*y)
    return Jump1
#-----#
#Here we compute the Product term in the integral part #
#-----#
def Betahat(XX,L):
    Betahat = []
    for i in range(n):
        V = np.ones(len(XX),dtype='float64')
        V = np.float64(V)
        p = i+1
        for l in range(p,n-1):
            U=(delta1[l]*L[l,i])*(np.exp(lambda_values[l]*XX)-1)/(1+delta1[l]*L[l,i])
            V=V*U+1
        Betahat.append(V)
        # print type(V)
    return Betahat
#-----#
#Here we computes the Levy Measure #
#-----#
def Mes(XX):
    M = []
    for i in range(n):
        w = alpha*np.abs(XX)
        from scipy.special import kn
        y = kv(1,w)
        t = (delta*alpha)/(np.pi*np.abs(XX))*np.exp(beta*np.abs(XX))*y
        M.append(t)
    return M
#-----#
#Here we computes the Intergral term in the drift #
#-----#
def II(L):
    XX =np.arange(0.05,100,0.5,dtype='float64') #Limit of integral from -INF TO INF
    YY =np.arange(-100,-0.05,0.5,dtype='float64')
    # XX =np.float64(XX)
    # YY =np.float64(YY)
    Res = []
    for i in range(n):
        G1 = ((np.exp(lambda_values[i]*XX)-1)*Betahat(XX,L)[i] - lambda_values[i]*XX)
        G2 = ((np.exp(lambda_values[i]*YY)-1)*Betahat(YY,L)[i] - lambda_values[i]*YY)
        Res.append((sum(G1*Mes(XX)[i])+sum(G2*Mes(YY)[i]))*0.5)
        # print G1,G2
    return Res
if __name__ == '__main__':

```

---

```

n = 4
dT = 0.5
sigma = 0.15
mean = 0
beta = 0
delta = 1.5
alpha = 1.5
c = np.sqrt(0.2)
a = 1
b = delta*np.sqrt(alpha**2 - beta**2)
W = np.zeros(n+1)
B = np.zeros((n+1,n+1))
T = np.arange(0, (n+1) * dT, dT)
L = np.zeros([n+1, n+1])
L[:,0] = 0.055 * np.ones(n+1)
delta1= 1.5*np.ones(n+1)
#The Lambda_values as initial values
lambda_values = np.arange(0.12,0.16,0.01)
R=list(lambda_values)
R.reverse()
lambda_values = np.array(R)
#-----#
#Here we comput diffH = NIG = N(alpha,beta,delta,mean) #
#-----#
for k in range(n):
    W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
    IGP1 = IGP(a,b,n)[1]
    diff = [IGP1[i] - IGP1[i-1] for i in range(1, len(IGP1))]
    for i in range(1,n+1):
        W[i] = W[i-1] + np.sqrt(diff[i-1])*random.gauss(0,1)
    H = np.zeros(len(IGP1))
    for i in range(len(IGP1)):
        H[i] = beta*delta**2*IGP1[i] + delta*W[i]
    diffH = [H[i+1]-H[i] for i in range(len(H)-1)]
#Call the Jump part function
Jump1 = jumppart(lambda_values, diffH)
#-----#
#Here we Simulate the Levy LIBOR Model Without Approximation #
#-----#
for col in range(n+1):
    for row in range(col+1,n+1):
        dummy = 0
        for k in range(row+1,n+1):
            dummy += (delta1[k]*L[k,col])/(1+delta1[k]*L[k,col])*lambda_values[k]
#Calculating LIBOR Rates
L[row][col+1] = L[row][col]*np.exp((-0.5*lambda_values[col]**2*c
-c*lambda_values[col]*dummy -II(L)[col])*dT + Jump1[col])

```

---

```
#-----#
#Here we plot the Levy LIBOR Rates                                     #
#-----#
    plt.figure()
    plt.plot(T, L[-1,:], color = 'red', label='Terminal LIBOR')
    plt.xlabel('Time')
    plt.ylabel('LIBOR Rate')
    plt.title('Terminal Levy LIBOR: Without Approximation')
    plt.legend()
    plt.show()
```

# Appendix G. Python Program For Picard Approximation of Lévy LIBOR: Using Numerical Integration

Here we show the simulation of Lévy LIBOR model with Picard approximation and Numerical integration of the product terms. The main equation for the approximate LIBOR used here is (8.1.5) in chapter 8. Note we consider the second order Picard approximation of the Lévy LIBOR model.

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as int
from scipy.special import kn
from sympy import *
import itertools
import random
#-----#
#This function computes the Inverse Guassian Random Random variable #
#-----#
def IGR(a, b, n):
    IGR = []
    while len(IGR) < n:
        v = random.gauss(0,1)
        u = random.uniform(0,1)
        y = v**2
        x = (a/b) + y/(2*b**2) + np.sqrt(4*a*b*y + y**2)/(2*b**2)
        if u <= a/(a+x*b):
            IGR.append(x)
        else: IGR.append(a**2/(b**2*x))
    return IGR
#-----#
#This function computes the Inverse Guassian Process #
#-----#
def IGP(a, b, n):
    dT = 0.5
    x = np.zeros(n+1)
    B = np.zeros((n+1,n+1))
    T = np.arange(0, 1 + dT, dT)
    v = IGR(a*dT,b,n+1)
    for i in range(1,n+1):
        x[i] = x[i-1] + v[i]
    return T, x
#-----#
```

---

```

#This function computes the the integral of the NIG process and the lambda_values generated #
#-----#

def jumppart1(lamda_values, diffH):
    Jump1 = []
    for x, y in itertools.izip(lamda_values, diffH):
        Jump1.append(x*y)
    return Jump1

#-----#
#   This function computes the the sum term                                     #
#-----#

def DeltaSum(ZZ):
    Dsum=[]
    for i in range(n-1):
        sumValue = 0
        p = i+1
        #print ZZ
        for l in range(p,n-1):
            sumValue+=((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*lambda_values[l]
        Dsum.append(sumValue)
    return Dsum

#-----#
#   This function computes the Beta Hat Term                                     #
#-----#

def Betahat(XX,ZZ):
    Betahat = []
    for i in range(n-1):
        V =np.ones(len(XX),dtype='float64')
        #V = np.dtype(float64)
        V = np.float64(V)
        p = i+1
        for l in range(p,n-1):
            U=(delta1[l]*np.exp(ZZ[l]))*(np.exp(lambda_values[l]*XX)-1)/(1+delta1[l]
            *np.exp(ZZ[l]))
            V=V*U+1
        Betahat.append(V)
        # print type(V)
    return Betahat

#-----#
#Here we computes the Levy Measure                                             #
#-----#

def Mes(XX):
    M = []
    for i in range(n-1):
        w = alpha*np.abs(XX)
        from scipy.special import kn

```

---

```

        y = kv(1,w)
        t = (delta*alpha)/(np.pi*np.abs(XX))*np.exp(beta*np.abs(XX))*y
        M.append(t)
    return M

#-----#
#   This function computes the Kumulant   first order   #
#-----#
def KDeltaSum(ZZ):
    Ksum=[]
    for i in range(n-1):
        KValue = 0
        p = i+1
        for l in range(p,n-1):
            KValue+= ((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l]))) \
                *(K(lambda_values[i]+ lambda_values[l] - K(lambda_values[i])-K(lambda_values[l])))
            A = K(lambda_values[i]) + KValue
            Ksum.append(A)
    return Ksum

#-----#
#   This function for the Drift Term called   b(s,T_{i},Z(1))   #

def driftTerm(ZZ):
    drift = []
    for i in range(len(lambda_values)):
        drift.append((-0.5*lambda_values[i]**2*c -c*lambda_values[i]*DeltaSum(ZZ)[i]
            -KDeltaSum(ZZ)[i])
    return drift
#print driftTerm(N,ZZ)

#-----#
# This is Where the main Program starts #

if __name__ == '__main__':
    n = 30
    dT = 0.5
    sigma = 0.25
    mean = 0
    beta = 0
    delta = 4.4
    alpha = 4.4
    c =np.sqrt(0.2)
    a = 1
    b = delta*np.sqrt(alpha**2 - beta**2)
    W = np.zeros(n+1)
    T = np.arange(0, (n-1) * dT, dT)
    #Lambda Values

```

---

```

lambda_values = np.arange(0.01,0.3,0.01)
L=list(lambda_values)
L.reverse()
lambda_values = np.array(L)
#Delta Constant Values

delta1= 1.5*np.ones(n)
#-----#
#Here we compute the Normal Inverse Gaussian Process                                     #
#-----#
for k in range(n):
    W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
IGP1 = IGP(a,b,n)[1]
diff = [IGP1[i] - IGP1[i-1] for i in range(1, len(IGP1))]
for i in range(1,n+1):
    W[i] = W[i-1] + np.sqrt(diff[i-1])*random.gauss(0,1)
X = np.zeros(len(IGP1))
for i in range(len(IGP1)):
    H[i] = beta*delta**2*IGP1[i] + delta*W[i]
diffH = [X[i]-X[i+1] for i in range(len(X)-1)]      #diffH = NIG = N(alpha,beta,delta,mean)

L = []
Z = 0.055 * np.ones(n-1)
Jump1 = jumppart1(lambda_values, diffX)
drift0 = driftTerm(Z)
# Here we find Z(1) which is the second picard approximation
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift0[i] + Jump1[i]
##Redo for Zhat: We use the second Picard approximation in the model
drift1 = driftTerm(Z)
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift1[i] + Jump1[i]
#-----#
# Here we find the exponential of Z to find L                                         #
#-----#
for i in Z:
    L.append(np.exp(i))
plt.plot(T,L)
plt.show()

```

# Appendix H. Python Program For Picard Approximation of Lévy LIBOR: Cumulant Expansion

Here we consider the simulation of Lévy LIBOR Market model with Picard approximation and cumulant expansion. The first order and the second order cumulant expansions can be seen in equations (8.2.2) and (8.2.3).

```

from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as int
from scipy.special import kn
from sympy import *
import itertools
import random
#-----#
#This function computes the Inverse Guassian Random Random variable      #
#-----#
def IGR(a, b, n):
    IGR = []
    while len(IGR) < n:
        v = random.gauss(0,1)
        u = random.uniform(0,1)
        y = v**2
        x = (a/b) + y/(2*b**2) + np.sqrt(4*a*b*y + y**2)/(2*b**2)
        if u <= a/(a+x*b):
            IGR.append(x)
        else: IGR.append(a**2/(b**2*x))
    return IGR
#-----#
#This function computes the Inverse Guassian Process                      #
#-----#
def IGP(a, b, n):
    dT = 0.5
    x = np.zeros(n+1)
    v = IGR(a*dT,b,n+1)
    for i in range(1,n+1):
        x[i] = x[i-1] + v[i]
    return x
#-----#
#This function computes the the integral of the NIG process and the lambda_values generated#
#-----#
def jumppart(lamda_values, X):
    Jump =[]

```



---

```

    for i in lambda_values:
        I=[]
        for j in X:
            I.append(i*j)
        Jump.append(sum(I))
    return Jump

def jumppart1(lamda_values, diffH):
    Jump1 =[]
    for x, y in itertools.izip(lamda_values, diffH):
        Jump1.append(x*y)
    return Jump1

#-----#
#   This function computes the the sum term                               #
#-----#
def DeltaSum(ZZ):
    Dsum=[]
    for i in range(n-1):
        sumValue = 0
        p = i+1
        for l in range(p,n-1):
            sumValue+=((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*
                lambda_values[l]
        Dsum.append(sumValue)
    return Dsum

#-----#
#   This function computes the Kumulant                                   #
#-----#
def K(lambda1):
    return delta*alpha - delta*np.sqrt(alpha**2-lambda1**2)

#-----#
#   This function computes the Kumulant  first order                     #
#-----#
def KDeltaSum(ZZ):
    Ksum=[]
    for i in range(n-1):
        KValue = 0
        p = i+1
        for l in range(p,n-1):
            KValue+= ((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*
                (K(lambda_values[i] +lambda_values[l] - K(lambda_values[i])
                -K(lambda_values[l])))
        A = K(lambda_values[i]) + KValue
        Ksum.append(A)
    return Ksum

#-----#
#   This function computes the Kumulant  Second order                     #
#-----#

```

```

#-----#
def KDeltaSum2(ZZ):
    Ksum2=[]
    for i in range(n-1):
        KValue = 0
        p = i+1
        for l in range(p,n-1): #for k in range(p1,n-1):
            KValue+= ((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))* \
                (K(lambda_values[i] +lambda_values[l]) + K(lambda_values[i])+ \
                K(lambda_values[l]))
            H = 0
            for k in range(p,l):
                H+=((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*((delta1[k] \
                *np.exp(ZZ[k]))/(1+delta1[k]*np.exp(ZZ[k])))*(K(lambda_values[i] \
                +lambda_values[l] +lambda_values[k])- \
                K(lambda_values[i]+ lambda_values[l])- \
                K(lambda_values[i]+ lambda_values[k])- \
                K(lambda_values[k]+lambda_values[l]) \
                + K(lambda_values[i]) + K(lambda_values[l]) \
                + K(lambda_values[k]))
            KValue+=H
        A = K(lambda_values[i]) + KValue
        Ksum2.append(A)
    return Ksum2
#-----#
# This function for the Drift Term called b(s,T_{i},Z(1))
def driftTerm(ZZ):
    drift = []
    for i in range(len(lambda_values)):
        drift.append(-0.5*lambda_values[i]**2*c -c*lambda_values[i]*DeltaSum(ZZ)[i]
        -KDeltaSum2(ZZ)[i])
    return drift
#-----#
# This is Where the main Program starts
if __name__ =='__main__':
    n = 10
    dT = 0.5
    mean = 0
    beta = 0
    delta = 1.5
    alpha = 1.5
    c =np.sqrt(0.2)
    a = 1
    b = delta*np.sqrt(alpha**2 - beta**2)
    W = np.zeros(n+1)
    T = np.arange(0, (n) * dT, dT)
    #Lambda Values

```

---

```

lambda_values = np.arange(0.12,0.20,0.01)
L=list(lambda_values)
L.reverse()
lambda_values = np.array(L)
#Delta Constant Values

delta1= 0.5*np.ones(n)
#-----#
#Here we compute the Normal Inverse Gaussian Process                                     #
#-----#
for k in range(n):
    W[k+1]= W[k] + np.sqrt(dT)*random.gauss(0,1)
IGP = IGP(a,b,n)
diff = [IGP[i] - IGP[i-1] for i in range(1, len(IGP))]
for i in range(1,n):
    W[i] = W[i-1] + np.sqrt(diff[i])*random.gauss(0,1)
H = np.zeros(len(IGP)-1)
for i in range(len(IGP)-1):
    H[i] = beta*delta**2*IGP[i] + delta*W[i]
diffH = [H[i+1]-H[i] for i in range(len(H)-1)] #dH = NIG = N(alpha,beta,delta,mean)
#Initial Z values :Z = Log(L)
Z = np.log(0.05)*np.ones(n-1)
Jump1 = jumppart1(lambda_values, diffH)
Jump2 = jumppart(lambda_values, diffH)
drift0 = driftTerm(Z)
# Here we find Z(1) which is the second picard approximation
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift0[i] + Jump1[i]
#Redo for Zhat: We use the second Picard approximation in the model
drift1 = driftTerm(Z)
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift1[i] + Jump1[i]
#Z.append(0.05) add the initial value to Z
#-----#
# Here we find the exponential of Z to find L                                         #
for i in Z:
    L.append(np.exp(i))
L.insert(0,IL)
#-----#
#Here we plot the Levy LIBOR Rates with Cumulant Expansion                         #
#-----#
plt.plot(T,L,label='Levy LIBOR Rate')
plt.title('Full Numerical Approximate Solution:Using First Order Cumulant')
plt.ylabel('LIBOR Rate')
plt.xlabel('Time')
plt.show()

```

# Appendix I. Python Program For Lévy LIBOR Model with Approximations using continuous jump (Brownian Motion)

The python code below shows the the simulation of the Lévy LIBOR Model with approximations but with Brownian motion as the jump component instead of the pure jump NIG process considered.

```
from __future__ import division
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate as int
from scipy.special import kn
from sympy import *
import itertools
import random

#-----#
#   This function computes the the sum term                                     #
#-----#
def DeltaSum(ZZ):
    Dsum=[]
    for i in range(n-1):
        sumValue = 0
        p = i+1
        for l in range(p,n-1):
            sumValue+=((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*
            lambda_values[l]
        Dsum.append(sumValue)
    return Dsum

#-----#
#   This function computes the Kumulant                                       #
#-----#
def K(lambda1):
    return delta*alpha - delta*np.sqrt(alpha**2-lambda1**2)

#-----#
#   This function computes the Kumulant first order                         #
#-----#
def KDeltaSum(ZZ):
    Ksum=[]
    for i in range(n-1):
        KValue = 0
        p = i+1
        for l in range(p,n-1):
            KValue+= ((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*
            (K(lambda_values[i] +lambda_values[l] - K(lambda_values[i]))-
```

---

```

        K(lambda_values[l]))
    A = K(lambda_values[i]) + KValue
    Ksum.append(A)
    return Ksum
#-----#
#   This function computes the Kumulant   Second order   #
#-----#
def KDeltaSum2(ZZ):
    Ksum2=[]
    for i in range(n-1):
        KValue = 0
        p = i+1
        for l in range(p,n-1): #for k in range(p1,n-1):
            KValue+= ((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))* \
                (K(lambda_values[i] +lambda_values[l]) + K(lambda_values[i])+ \
                K(lambda_values[l]))
            H = 0
            for k in range(p,l):
                H+=((delta1[l]*np.exp(ZZ[l]))/(1+delta1[l]*np.exp(ZZ[l])))*\
                    ((delta1[k]*np.exp(ZZ[k]))/(1+delta1[k]*np.exp(ZZ[k])))* \
                    (K(lambda_values[i]+lambda_values[l] + lambda_values[k])- \
                    K(lambda_values[i]+lambda_values[l])- K(lambda_values[i]+ lambda_values[k]) \
                    -K(lambda_values[k]+ lambda_values[l]) + K(lambda_values[i]) + K(lambda_values[l])+ \
                    K(lambda_values[k]))
                KValue+=H
            A = K(lambda_values[i]) + KValue
            Ksum2.append(A)
    return Ksum2
#-----#
#   This function for the Drift Term called   b(s,T_{i},Z(1))
def driftTerm(ZZ):
    drift = []
    for i in range(len(lambda_values)):
        drift.append((-0.5*lambda_values[i]**2*c -c*lambda_values[i]*DeltaSum(ZZ)[i]
        -KDeltaSum2(ZZ)[i])
    return drift
#-----#
# This is Where the main Program starts
if __name__ == '__main__':
    n = 10
    dT = 0.5
    mean = 0
    beta = 0
    delta = 1.5
    alpha = 1.5
    c =np.sqrt(0.2)
    a = 1
    b = delta*np.sqrt(alpha**2 - beta**2)

```

---

```

W1 = np.zeros(n+1)
T = np.arange(0, (n) * dT, dT)
#Lambda Values

lambda_values = np.arange(0.12,0.20,0.01)
L=list(lambda_values)
L.reverse()
lambda_values = np.array(L)
#Delta Constant Values

delta1= 0.5*np.ones(n)
#-----#
#Here we compute the Normal Inverse Gaussian Process                                     #
#-----#
#The Brownian motion used as the jump term
for i in range(1,n):
    W1[i] = W1[i-1] + np.sqrt(diff[i])*random.gauss(0,1)

#Initial Z values :Z = Log(L)
Z = np.log(0.05)*np.ones(n-1)
drift0 = driftTerm(Z)
# Here we find Z(1) which is the second picard approximation
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift0[i] + W1[i]
#Redo for Zhat: We use the second Picard approximation in the model
drift1 = driftTerm(Z)
for i in range(len(lambda_values)):
    Z[i] = Z[i] + drift1[i] + W1[i]
#Z.append(0.05) add the initial value to Z
#-----#
# Here we find the exponential of Z to find L                                           #
# for i in Z:                                                                           #
#     L.append(np.exp(i))
#     L.insert(0,IL)
#-----#
#Here we plot the Levy LIBOR Rates with Cumulant Expansion                             #
#-----#

plt.plot(T,L,label='Levy LIBOR Rate')
plt.title('Full Numerical Approximate Solution:Using First Order Cumulant')
plt.ylabel('LIBOR Rate')
plt.xlabel('Time')
plt.show()

```

---

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